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Group rings that are exact

Yaroslav Shitov

National Research University Higher School of Economics, 20 Myasnitskaya Ulitsa, Moscow 101000, Russia

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ABSTRACT

A ring R is left exact if, for every finitely generated left submodule $S \subset R^n$, every left R-linear function from S to R extends to a left R-linear function from R^n to R. The class of exact rings generalizes that of self-injective rings and has been introduced in a recent paper by Wilding, Johnson, and Kambites. In our paper we show that the group ring of a group G over a ring R is left exact if and only if R is left exact and G is locally finite.

1. Introduction

The word *ring* will stand throughout our paper for an associative ring with a unity (which is different from zero), and R will denote such a ring. Recall that R is called *left self-injective* if it is injective as a left module over itself, that is, any left R-linear function from a submodule X of a left R-module Y to R can be extended to a left R-linear function from Y to R.

In a recent paper [3] Wilding, Johnson, and Kambites introduced a more general class of rings which they called *exact* rings. For any m-by-n matrix A over R, they define the left R-module $\mathrm{Row}(A)$ as the set of all vectors $x \in R^{1\times n}$ that can be written as

E-mail address: yaroslav-shitov@yandex.ru.

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$x = uA$ for some vector $u \in R^{1\times m}$. The right R-module $\mathrm{Col}(A)$ is defined as the set of all vectors $y \in R^{m\times 1}$ that have the form $y = Av$ for some $v \in R^{n\times 1}$. As Theorem 3.4 of [3] shows, the ring R is *exact* if and only if, for every matrix $A \in R^{m\times n}$, every left R-linear function from $\mathrm{Row}(A)$ to R extends to a left R-linear function from $R^{1\times n}$ to R, and every right R-linear function from $\mathrm{Col}(A)$ to R extends to a right R-linear function from $R^{m\times 1}$ to R.

In our paper, we will separate between the concepts of left and right exactness. Noting that a left (right, respectively) submodule $S \subset R^n$ is finitely generated if and only if $S = \mathrm{Row}(A)$ ($S = \mathrm{Col}(A)$, respectively) for some matrix A over R, we can rewrite the definition of exactness as follows.

Definition 1.1. A ring R is *left exact* if, for every finitely generated left submodule $S \subset R^n$ and every left R-linear function $\varphi : S \to R$, there is a left R-linear function $\psi : R^n \to R$ satisfying $\varphi(x) = \psi(x)$ for any $x \in S$.

Replacing the word 'left' by 'right' everywhere in Definition 1.1, we obtain the definition of right exactness. As noted above, the authors of [3] call a ring exact if and only if is both left and right exact in our notation. Similarly, we will call a ring *self-injective* if it is both left and right self-injective.

As Definition 1.1 shows, we can get the definition of exactness from that of self-injectivity by requiring X to be a finitely generated submodule of a free module $Y = R^n$. So any left self-injective ring must also be left exact, and in [3] the question has been asked whether the converse is true or not.

Question 1.2. (See [3, Section 8].) Does there exist a ring which is exact but not self-injective?

In our paper we examine the concept of exactness by applying it to studying the group rings. Recall that the group ring of a group G over R is the set $R[G]$ of all formal sums $\rho = \sum_{h\in G} \rho_h\, h$ whose *coefficients* ρ_h belong to R and whose *support* (the set of all $h \in G$ satisfying $\rho_h \neq 0$) is finite. The sum $\rho' + \rho''$ is defined by $[\rho' + \rho'']_h = \rho'_h + \rho''_h$, and the product $\rho'\rho''$ by $[\rho'\rho'']_h = \sum_{g\in G} \rho'_{g^{-1}}\rho''_{gh}$. An element $\rho_e \in R$ is called the *constant term* of ρ if e is the unity of G, and we identify the subring of $R[G]$ formed by the constant terms with the ring R.

The goal of our paper is to obtain a characterization of exact group rings similar to the following which is known for self-injective rings.

Theorem 1.3. *(See [1].) Let $\mathbb{F}$ be a field and G a group. The ring $\mathbb{F}[G]$ is left self-injective if and only if G is finite.*

Namely, the main result of our paper will state that $R[G]$ is left exact if and only if R is left exact and G is locally finite. This result will imply a positive answer for