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Invariants of algebraic derivations and automorphisms in Banach algebras

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ABSTRACT

We prove that if a semisimple real or complex Banach algebra A possesses an algebraic derivation whose invariants are algebraic, then A is finite-dimensional. This result is a full generalization of a recent result by Haily, Kaidi and Palacios (2011) [15] for the case of inner derivations in complex semisimple Banach algebras. The analogous result for automorphism case is also obtained.

1. Introduction

Let A be an associative algebra over a field $\mathbb{F}$. By $Z(A)$ and $J(A)$ we denote the center of A and the Jacobson radical of A, respectively. We say that A is semisimple if $J(A) = 0$. An element $b \in A$ is called algebraic over $\mathbb{F}$ if there exists a nonzero polynomial $f(x) \in \mathbb{F}[x]$ such that $f(b) = 0$. A subset of A is said to be algebraic if all its elements are algebraic. For $b \in A$, the centralizer of b in A is the subalgebra $C_A(b) = \{a \in A \mid ab = ba\}$.

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A linear map $d : A \to A$ is called a derivation of A if $d(ab) = d(a)b + ad(b)$ for all $a, b \in A$. Those $a \in A$ with $d(a) = 0$ are called invariants (or constants) of d. We set

$$A^{(d)} = \{a \in A \mid d(a) = 0\},$$

the subalgebra of invariants of d in A. d is called algebraic over $\mathbb{F}$ if there exist an integer $n \geqslant 1$ and $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{F}$ such that $d^n(a) + \alpha_1 d^{n-1}(a) + \cdots + \alpha_{n-1}d(a) = 0$ for all $a \in A$. For $b \in A$, the map $\mathrm{ad}(b) : a \in A \mapsto ba - ab$ defines a derivation of A, called the inner derivation of A defined by b. Clearly, if $d = \mathrm{ad}(b)$ is an inner derivation, then $A^{(\mathrm{ad}(b))} = C_A(b)$.

Several authors have studied the structure of centralizers and the information that they can provide on the whole algebra. Herstein and Neumann [17] initiated this line of research by proving that the simplicity of $C_A(b)$ implies the simplicity of A if A is semiprime and $b \in A$ is integral over its center. Under the assumption that A is semiprime and $b \in A$ is algebraic, Cohen [11] proved that if $C_A(b)$ is semiprime artinian, then so is A. Some other related results can be found in [23,27,29]. Recall that the subalgebra of invariants of the inner derivation defined by $b \in A$ equals the centralizer of b in A. More generally, the subalgebras of invariants under the action of a single derivation or finite-dimensional Lie algebras of derivations of associative algebras have been extensively investigated by many authors. In particular, the relation between various finiteness conditions of algebras and their subalgebras of invariants have been studied (see [1–4,9,10,13,14] for instance). Recently, Haily, Kaidi and Palacios [15] gave an affirmative answer to [8, Question 2] and proved that a semisimple complex unital Banach algebra A contains an element whose centralizer is algebraic, then A is finite-dimensional. This result can be reformulated as follows:

Theorem 1.1. *(See [15, Theorem 2.3].) Suppose A is a semisimple complex unital Banach algebra containing an element $b \in A$ such that $A^{(\mathrm{ad}(b))}$ is algebraic. Then A is finite-dimensional.*

Let σ be a linear automorphism of A. σ is called algebraic over $\mathbb{F}$ if there exist an integer $n \geqslant 1$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ such that $\sigma^n(a) + \alpha_1\sigma^{n-1}(a) + \cdots + \alpha_{n-1}\sigma(a) + \alpha_n a = 0$ for all $a \in A$. Those $a \in A$ with $\sigma(a) = a$ are called invariants (or fixed points) of σ. We set $A^{(\sigma)} = \{a \in A \mid \sigma(a) = a\}$, the subalgebra of invariants of σ in A. Several results on the invariants of automorphisms can be found in [3–5,21,24,25]. For a unit $u \in A$, the map $\sigma_u : a \in A \mapsto uau^{-1}$ defines a linear automorphism of A, called the inner automorphism of A defined by u. Note that $A^{(\sigma_u)} = A^{(\mathrm{ad}(u))}$. As an immediate consequence of Theorem 1.1, we have

Theorem 1.2. *Suppose A is a semisimple complex unital Banach algebra containing a unit element $u \in A$ such that $A^{(\sigma_u)}$ is algebraic. Then A is finite-dimensional.*