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Dimension of zero weight space: An algebro-geometric approach

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ABSTRACT

Let G be a connected, adjoint, simple algebraic group over the complex numbers with a maximal torus T and a Borel subgroup B containing T. The study of zero weight spaces in irreducible representations of G has been a topic of considerable interest; there are many works which study the zero weight space as a representation space for the Weyl group. In this paper, we study the variation on the dimension of the zero weight space as the highest weight of the irreducible representation varies over the set of dominant integral weights of T, which are lattice points in a certain polyhedral cone. The theorem proved here asserts that the zero weight spaces have dimensions which are piecewise quasi-polynomial functions on the polyhedral cone of dominant integral weights.The main tool we use are the Geometric Invariant Theory and the Riemann–Roch theorem.

1. Introduction

Let G be a connected, adjoint, simple algebraic group over the complex numbers $\mathbb{C}$ with a maximal torus T and a Borel subgroup $B \supset T$. The study of zero weight spaces in irreducible representations of G has been a topic of considerable interest; there are many works which study the zero weight space as a representation space for the Weyl group. In this paper, we study the variation of the dimension of the zero weight space

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as the irreducible representation varies over the set of dominant integral weights for T, which are lattice points in a certain polyhedral cone (using algebro-geometric methods).

The theorem proved here asserts that the zero weight spaces have dimensions which are piecewise polynomial functions on the polyhedral cone generated by dominant integral weights. The precise statement of the theorem is given below.

Let $\Lambda = \Lambda(T)$ be the character group of T and let $\Lambda^+ \subset \Lambda$ (resp. Λ^{++}) be the semigroup of dominant (resp. dominant regular) weights. Then, by taking derivatives, we can identify Λ with Q, where Q is the root lattice (since G is an adjoint group). For $\lambda \in \Lambda^+$, let $V(\lambda)$ be the irreducible G-module with highest weight λ. Let $\mu_0 : \Lambda^+ \to \mathbb{Z}_+$ be the function: $\mu_0 = \dim V(\lambda)_0$, where $V(\lambda)_0$ is the 0-weight space of $V(\lambda)$.

Let $\Gamma = \Gamma_G \subset Q$ be the sublattice as in Theorem 3.1.

Also, let $\Lambda(\mathbb{R}) := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ and let $\Lambda^+(\mathbb{R})$ be the cone inside $\Lambda(\mathbb{R})$ generated by Λ^+. Further, let $\Lambda^{++}(\mathbb{R})$ be the relative interior of $\Lambda^+(\mathbb{R})$. Let $C_1, \ldots, C_N \subset \Lambda^{++}(\mathbb{R})$ be the chambers (i.e., the GIT classes in $\Lambda^{++}(\mathbb{R})$ of maximal dimension: equal to the dimension of $\Lambda(\mathbb{R})$, with respect to the T-action) (see Section 2).

For any $w \in W$ and $1 \leqslant i \leqslant \ell$, define the hyperplane

$$H_{w,i} := \bigl\{\lambda \in \Lambda(\mathbb{R}):\ \lambda(wx_i) = 0\bigr\},$$

where W is the Weyl group of G and $\{x_1, \ldots, x_\ell\}$ is the basis of $\mathfrak{t}$ dual to the basis of $\mathfrak{t}^*$ given by the simple roots. Then, by virtue of Corollary 3.6, $C_1, \ldots, C_N$ are the connected components of

$$\Lambda^{++}(\mathbb{R}) \setminus \Bigl(\bigcup_{w\in W,\, 1\leqslant i\leqslant \ell} H_{w,i}\Bigr).$$

With this notation, we have the following main result of our paper (cf. Theorem 4.1).

Theorem 1.1. *Let $\bar{\mu} = \mu + \Gamma$ be a coset of Γ in Q. Then, for any GIT class C_k, $1 \leqslant k \leqslant N$, there exists a polynomial $f_{\bar{\mu},k} : \Lambda(\mathbb{R}) \to \mathbb{R}$ with rational coefficients of degree $\leqslant \dim_{\mathbb{C}} X - \ell$, such that*

$$f_{\bar{\mu},k}(\lambda) = \mu_0(\lambda), \quad \textit{for all } \lambda \in \bar{C}_k \cap \bar{\mu}, \tag{1}$$

where $\bar{C}_k$ is the closure of C_k inside $\Lambda(\mathbb{R})$ and X is the full flag variety G/B. Further, $f_{\Gamma,k}$ has constant term 1.

The proof of the above theorem relies on Geometric Invariant Theory (GIT). Specifically, we realize the function μ_0 restricted to $\bar{C}_k \cap \Lambda$ as an Euler–Poincaré characteristic of a reflexive sheaf on a certain GIT quotient (depending on C_k) of $X = G/B$ via the maximal torus T. Then, one can use the Riemann–Roch theorem for singular varieties to calculate this Euler–Poincaré characteristic. From this calculation, we conclude that