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On boundedly generated subgroups of profinite groups

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ABSTRACT

In this paper we investigate the following general problem. Let G be a group and let $i(G)$ be a property of G. Is there an integer d such that G contains a d-generated subgroup H with $i(H) = i(G)$? Here we consider the case where G is a profinite group and H is a closed subgroup, extending earlier work of Lucchini and others on finite groups. For example, we prove that $d = 3$ if $i(G)$ is the prime graph of G, which is best possible, and we show that $d = 2$ if $i(G)$ is the exponent of a finitely generated prosupersolvable group G.

1. Introduction

Let G be a finite group and let $i(G)$ be a group invariant. For example, $i(G)$ could be the set of composition factors of G, or the exponent of G, or the set of prime divisors of $|G|$, and so on. Given such a property, one can ask whether or not it can be detected from subgroups of G generated by very few elements. For instance, a well-known theorem of Thompson [24] states that G is solvable if and only if every 2-generated subgroup is solvable.

Let $d \in \mathbb{N}$ be minimal such that G contains a d-generated subgroup H with $i(H) = i(G)$. This integer is studied by Lucchini, Morigi and Shumyatsky in [21], where several interesting results are established. For example, they prove that $d = 2$ if $i(G) = \pi(G)$ is

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the set of prime divisors of $|G|$ (cf. Problem 17.125 in [15]), and $d = 3$ if $i(G) = \Gamma(G)$ is the *prime graph* of G, which is a graph with vertex set $\pi(G)$, and two vertices p and q are adjacent if and only if G contains an element of order pq. They also show that $d \leqslant 4$ if $i(G) = \exp(G)$ is the exponent of G, and the better bound $d \leqslant 3$ has recently been established by Detomi and Lucchini [7, Theorem 1.6] (determining whether or not $d = 2$ in this situation is an open problem).

The main aim of this paper is to extend the study of boundedly generated subgroups initiated in [21] from finite groups to *profinite* groups, with some suitable (and necessary) modifications.

Recall that a profinite group is a topological group which is an inverse limit of finite groups (which are equipped with the discrete topology). Given a closed subgroup H of a profinite group G, its index $|G : H|$ is the least common multiple of the indices of the open subgroups of G containing H. Hence, the order of a profinite group G is defined to be $|G : 1|$, which is a *supernatural number* (or *Steinitz number*), that is, a formal infinite product $\prod p^{n(p)}$ over all primes p, in which each $n(p)$ is a non-negative integer or infinity. In addition, the order of an element $g \in G$, denoted by $|g|$, is defined to be the order of the subgroup topologically generated by g, that is, $|g| = |\overline{\langle g\rangle}|$. If d is a positive integer then a closed subgroup H of G is said to be d-generated if $H = \overline{\langle x_1, \dots, x_d\rangle}$ for some $x_i \in H$, which is equivalent to the condition that $HN/N = \langle x_1N, \dots, x_dN\rangle$ for every open normal subgroup N of G.

Given these definitions, we can consider the exponent of a profinite group G and the set of prime divisors of $|G|$, denoted by $\exp(G)$ and $\pi(G)$, respectively. We can also define the prime graph $\Gamma(G)$.

Our first result is a natural extension of [21, Theorem C].

Theorem A. *Let G be a profinite group. Then there exists a 3-generated (closed) subgroup H of G such that $\Gamma(H) = \Gamma(G)$.*

This result is best possible. Indeed, there exists a finite group G such that $\Gamma(H) \neq \Gamma(G)$ for every 2-generated subgroup H of G (see [21, p. 883]). A key tool in the proof of Theorem A is an extension of [21, Theorem C] for finite groups (see Proposition 3.1): if $1 = M_n \leqslant \cdots \leqslant M_0 = M$ is a normal series of a finite group M, then there is a 3-generated subgroup K of M such that $\Gamma(KM_i/M_i) = \Gamma(M/M_i)$ for all i.

In order to prove Proposition 3.1, we will show that if M has no proper subgroup K with the desired property, then M is 3-generated (so the conclusion holds with $K = M$). In particular, we are naturally led to consider the minimum number of generators of a finite group. A fundamental role in this investigation is played by the so-called *crown-based powers* of a monolithic group (a finite group is said to be *monolithic* if it has only one minimal normal subgroup). The notion of a *crown* was introduced by Gaschütz [9] in the context of solvable groups, in his construction of *prefrattini subgroups*. More recently, this notion has been generalized to all finite groups (see [8], for example). In [6], Detomi and Lucchini introduce crown-based powers as an extension of crowns, where