Journal of Algebra 406 (2014) 154–170

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Euclidean pairs and quasi-Euclidean rings

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ARTICLE INFO

Article history:
Received 9 October 2013
Available online 18 March 2014
Communicated by Louis Rowen

Keywords:
(Quasi-)Euclidean rings
Euclidean algorithm
K-Hermite rings
Bézout rings
Regular rings
GE-rings
Stable range
Singular matrices
Products of idempotents

ABSTRACT

We study the interplay between the classes of right quasi-Euclidean rings and right K-Hermite rings, and relate them to projective-free rings and Cohn's GE_2-rings using the method of noncommutative Euclidean divisions and matrix factorizations into idempotents. Right quasi-Euclidean rings are closed under matrix extensions, and a left quasi-Euclidean ring is right quasi-Euclidean if and only if it is right Bézout. Singular matrices over left and right quasi-Euclidean domains are shown to be products of idempotent matrices, generalizing an earlier result of Laffey for singular matrices over commutative Euclidean domains.

1. Introduction and definitions

Inspired by Howie's work [11] on idempotents in transformation semigroups of sets, J.A. Erdos [6] proved that singular matrices over a field can be decomposed as products of idempotent matrices. This was extended in different directions by several authors (e.g. [1,3,13,7,21]). The decomposition of 2×2 matrices with a zero row over a commutative Euclidean domain was one of the main steps in Laffey's work [13]. In connection with

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http://dx.doi.org/10.1016/j.jalgebra.2014.02.009
0021-8693/

this decomposition, we exploit the general notions of **right Euclidean pairs** and **right quasi-Euclidean rings** in this paper. This leads, in Section 2, to an easy and elementary proof for the idempotent decomposition of $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$ over a possibly noncommutative right quasi-Euclidean ring. After proving this result, we show in Sections 2–3 that right Bézout rings of stable range one are right quasi-Euclidean, and that a general ring is right quasi-Euclidean if and only if it is a right K-Hermite ring (right Hermite ring in the sense of I. Kaplansky [12]) and a GE_2-ring (in the sense of P.M. Cohn [4]). In Section 3, it is also proved that matrix rings over right quasi-Euclidean rings remain right quasi-Euclidean. (Thus, for instance, $\mathbb{M}_n(\mathbb{Z})$ and $\mathbb{M}_n(\mathbb{Q}[x])$ provide new noncommutative examples of right and left quasi-Euclidean rings.) However, an example of Bergman in Section 4 shows that left and right quasi-Euclidean regular rings need not be Dedekind-finite. In Section 5, we revisit the theme of idempotent factorization of matrices, and prove our last main result (Theorem 25) that, over any left and right quasi-Euclidean domain, singular matrices are products of idempotent matrices.

A (not necessarily commutative) integral domain R is called a *right Euclidean domain* if there is a map $\varphi : R \setminus \{0\} \to \{0, 1, 2, \ldots\}$ such that, for any $a, b \in R$ with $b \neq 0$, there exists an equation $a = bq + r$ in R where either $r = 0$, or $\varphi(r) < \varphi(b)$. A *right chain ring* is a ring whose right ideals form a chain under inclusion; or equivalently, for any $a, b \in R$, we have either $aR \subseteq bR$ or $bR \subseteq aR$. A ring R is called a right Bézout ring if each finitely generated right ideal is principal. (For instance, any right chain ring and any principal right ideal ring is right Bézout.) A ring R is called *right K-Hermite* (after Kaplansky [12], but following the terminological convention of [16, I.4.23]) if for every pair $(a, b) \in R^2$ there exist an element $r \in R$ and a matrix $Q \in \mathrm{GL}_2(R)$ such that $(a, b) = (r, 0)Q$. Kaplansky has shown in [12] that any right K-Hermite ring is right Bézout, and Amitsur has shown in [2] that the converse holds if the ring in question is an integral domain. Needless to say, similar definitions and remarks can be made when the adjective "right" is replaced by "left".

An ordered pair (a, b) over any ring R is said to be a *right Euclidean pair* if there exist elements $(q_1, r_1), \ldots, (q_{n+1}, r_{n+1}) \in R^2$ (for some $n \geqslant 0$) such that $a = bq_1 + r_1$, $b = r_1 q_2 + r_2$, and

$$r_{i-1} = r_i q_{i+1} + r_{i+1} \quad \text{for } 1 < i \leqslant n, \text{ with } r_{n+1} = 0. \qquad (*)$$

The notion of a *left Euclidean pair* is defined similarly. In the following, when we talk about "Euclidean pairs", we shall always mean *right* Euclidean pairs. If all pairs $(a, b) \in R^2$ are right Euclidean, we say that R is a *right quasi-Euclidean ring*. Clearly, factor rings and finite direct products of right quasi-Euclidean rings remain right quasi-Euclidean. For instance, right chain rings and factor rings of right Euclidean domains are right quasi-Euclidean rings. The notion of (right) quasi-Euclidean rings was introduced without a name by O'Meara [20], and with a name (and a somewhat different but equivalent definition) by Leutbecher [17]. More recently, Glivický and Šaroch [8] studied certain special classes of commutative quasi-Euclidean domains. In both [17] and [8],