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Isomorphic pairs of quadratic forms

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ABSTRACT

Let k be a field of characteristic distinct from 2, V a finite dimensional vector space over k. We call two pairs of quadratic k-forms (f_1, g_1), (f_2, g_2) on V isomorphic if there exists an isomorphism $s : V \to V$ such that $f_2 = f_1 \circ s$, $g_2 = g_1 \circ s$. We prove that if $f_1 + tg_1 \simeq f_2 + tg_2$ over $k(t)$ and either the form $f_1 + tg_1$ is anisotropic, or $\det(f_1 + tg_1)$ is a squarefree polynomial of degree at least $\dim V - 1$, then the pairs (f_1, g_1) and (f_2, g_2) are isomorphic.

Let k be a field of characteristic distinct from 2, V a finite dimensional vector space over k, $\dim V = n$. Let further (f_1, g_1), (f_2, g_2) be two pairs of quadratic k-forms on V. We call the pairs (f_1, g_1) and (f_2, g_2) isomorphic and denote this as $(f_1, g_1) \simeq (f_2, g_2)$, if there exists an isomorphism $s : V \to V$ such that $f_2 = f_1 \circ s$, $g_2 = g_1 \circ s$. If we fix a basis of V, then f_1, f_2 are just homogeneous quadratic polynomials, which, slightly abusing notation, we identify with the corresponding symmetric matrices. In this interpretation isomorphism of pairs means that there exists a matrix $S \in GL_n(k)$ (the matrix of the map s with respect to the given basis) such that $f_2 = S^t f_1 S$, $g_2 = S^t g_1 S$. For any pair of forms (f, g) we can consider $\det(f + tg)$ as an element of $k[t]$ modulo k^{*2}, t being an indeterminate. Clearly, if $(f_1, g_1) \simeq (f_2, g_2)$, then $f_1 + tg_1 \simeq f_2 + tg_2$ over $k(t)$ and $\det(f_1 + tg_1)k^{*2} = \det(f_2 + tg_2)k^{*2}$. In general the converse is not true. As a simple

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counterexample consider the forms $f_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $g_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $f_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $g_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Then $f_1 + tg_1 \simeq f_2 + tg_2 \simeq \mathbb{H}$, and $\det(f_1 + tg_1) = \det(f_2 + tg_2) = -1$, but, obviously, $g_1 \not\simeq g_2$, hence $(f_1, g_1) \not\simeq (f_2, g_2)$. The main purpose of this note is to obtain certain sufficient conditions, providing isomorphism of the pairs (f_1, g_1) and (f_2, g_2). In the following statement we give such a condition and make clear why in the counterexample above the form $f_1 + tg_1$ is isotropic.

Proposition 1. *Assume that $f_1 + tg_1 \simeq f_2 + tg_2$ and the form $f_1 + tg_1$ is anisotropic. Then $(f_1, g_1) \simeq (f_2, g_2)$.*

Proof. We recall the Amer theorem, which is the sticking point in the proof: *Let Q_1, Q_2 be quadratic forms on a finite dimensional vector space V over a field k. Then Q_1, Q_2 vanish on a common r-dimensional subspace of V if and only if $Q_1 + tQ_2$ vanishes on an r-dimensional subspace of $V_{k(t)}$* [1,3].

Since $f_1 + tg_1 \simeq f_2 + tg_2$, the form $(f_1 \perp -f_2) + t(g_1 \perp -g_2)$ is hyperbolic, i.e. $(f_1 \perp -f_2) + t(g_1 \perp -g_2) \simeq n\mathbb{H}$. Hence by the Amer theorem applied to the forms $f_1 \perp -f_2$ and $g_1 \perp -g_2$, there is an n-dimensional subspace of $V \oplus V$, generated by vectors (v_{i1}, v_{i2}) $(1 \leqslant i \leqslant n)$ such that

$$f_1\left(\sum_{i=1}^n x_i v_{i1}\right) = f_2\left(\sum_{i=1}^n x_i v_{i2}\right), \qquad g_1\left(\sum_{i=1}^n x_i v_{i1}\right) = g_2\left(\sum_{i=1}^n x_i v_{i2}\right)$$

for any coefficients $x_i \in k$. Suppose $\sum_{i=1}^n \lambda_i v_{i1} = 0$ for some $\lambda_i \in k$. If $\sum_{i=1}^n \lambda_i v_{i2} \neq 0$, then we have

$$f_2\left(\sum_{i=1}^n \lambda_i v_{i2}\right) = f_1\left(\sum_{i=1}^n \lambda_i v_{i1}\right) = 0, \qquad g_2\left(\sum_{i=1}^n \lambda_i v_{i2}\right) = g_1\left(\sum_{i=1}^n \lambda_i v_{i1}\right) = 0.$$

Hence the forms f_2 and g_2 have a common nontrivial zero, which implies that the form $f_2 + tg_2 \simeq f_1 + tg_1$ is isotropic, a contradiction. Therefore, $\sum_{i=1}^n \lambda_i v_{i2} = 0$, so $\sum_{i=1}^n \lambda_i (v_{i1}, v_{i2}) = 0$. Since the vectors (v_{i1}, v_{i2}) are linearly independent, we conclude that $\lambda_1 = \cdots = \lambda_n = 0$, which means that the set $\{v_{i1}\}$ is linearly independent. By symmetry the set $\{v_{i2}\}$ $(1 \leqslant i \leqslant n)$ is linearly independent as well. In other words, the sets $\{v_{i1}\}$ and $\{v_{i2}\}$ $(1 \leqslant i \leqslant n)$ are bases of V. Now let $s : V \to V$ be the isomorphism determined by $s(v_{i2}) = v_{i1}$. Then $f_2 = f_1 \circ s$, $g_2 = g_1 \circ s$, which implies that $(f_1, g_1) \simeq (f_2, g_2)$. □

Corollary 2. *Let k be a field,* $\operatorname{char} k \neq 2$, *$n$ a positive integer, $b_{ij} + c_{ij}t$ linear polynomials in $k[t]$, $1 \leqslant i, j \leqslant n$. Suppose that the $n \times n$ matrix $A = (b_{ij} + c_{ij}t)$ is symmetric, the corresponding quadratic form $f + tg$ over $k(t)$ is anisotropic, and* $\det A \in k$. *Then all the entries of A are elements of k, i.e. $c_{ij} = 0$ for any i, j.*