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Weakly clean rings

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ABSTRACT

We study the class of weakly clean rings which were introduced in [15]. It is known that weakly clean rings are a subclass of exchange rings and that they contain clean rings as a proper subclass. In this paper we prove that weakly clean rings also contain some other important examples of exchange rings, such as π-regular rings and C^*-algebras of real rank zero. Further, we prove that many classes of weakly clean rings can be viewed as corners of clean rings. This, for example, implies that every π-regular ring and every C^*-algebra of real rank zero is a corner of a clean ring. Lastly, we study the question when the ideal extension of weakly clean rings is weakly clean, and we give an example of a non-weakly clean exchange ring, answering the question in [15].

1. Introduction

An element of a ring is called *clean* if it is a sum of an idempotent and a unit, and a ring is *clean* if its every element is clean. This notion was introduced by Nicholson in [12] where it was proved that every clean ring is an exchange ring and if idempotents in the ring are central then the converse also holds (see [12, Proposition 1.8]). Other examples of clean rings include semiperfect rings [7], unit-regular rings [5] and endomorphism rings of continuous modules [6].

The class of *weakly clean rings* was introduced in [15]. A ring R is said to be *weakly clean* if for every $a \in R$ there exist an idempotent $e \in R$ and a unit $u \in R$ such that $a - e - u \in (1 - e)Ra$, or equivalently, the matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is a clean element of the ring

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$M_2(R)$ (see [15, Proposition 2.2]). As shown in [15], every clean ring is weakly clean and every weakly clean ring is exchange. The well-known example due to Bergman is an example of a non-clean weakly clean ring (see [15, Example 3.1]). The question whether there exists a non-weakly clean exchange ring was left open in [15].

In [15], the notion of weakly clean rings was introduced with the purpose of answering the question whether corners of clean rings are clean. In this paper we study weakly clean rings as a separate subject of interest. The motivation for our interest comes from the obtained results. For example, we show that the class of weakly clean rings includes some important examples of exchange rings, such as π-regular rings and C^*-algebras of real rank zero. Further, we show that the class of weakly clean rings is closely related to the class of corners of clean rings. For example, we prove that for algebras over a field, and for rings with a nonzero characteristic, these two classes actually coincide. This, for example, implies that a unital C^*-algebra has real rank zero if and only if it is a corner of a clean ring. Also, every π-regular ring is a corner of a clean ring.

In the last section we generalize the notion of the weakly clean property to non-unital rings. Inspired by Ara's extension theorem for exchange rings [1, Theorem 2.2], we prove that, under some additional hypotheses, an analogous theorem holds also for weakly clean rings. In particular, we show that if R is a ring with an ideal I such that R/I is regular, I is weakly clean and idempotents lift modulo I, then R is weakly clean (Theorem 4.5). Also, we prove that if I and R/I are both π-regular rings then R is weakly clean (Proposition 4.9). We provide some examples showing that these statements cannot be generalized much further. These examples, in particular, are examples of non-weakly clean exchange rings, and thus they give a negative answer to the question in [15], asking whether or not every exchange ring is weakly clean.

All rings in this paper will be non-commutative and unital, unless otherwise specified. Rings that do not necessarily have a unit will be called *non-unital*. For a ring R, we denote by $\mathrm{Id}(R)$, $U(R)$, $J(R)$ and $M_n(R)$ the set of idempotents, the set of units, the Jacobson radical and the ring of $n \times n$ matrices over R, respectively.

2. Properties and examples

First let us recall the definition of a weakly clean ring.

Definition 2.1. (See [15, Definition 2.3].) Let R be a ring. An element $a \in R$ is *weakly clean* in R if the following equivalent conditions hold:

(i) There exist $e \in \mathrm{Id}(R)$ and $u \in U(R)$ such that $a - e - u \in (1 - e)Ra$.
(ii) There exist an idempotent $e \in Ra$ and a unit $u \in U(R)$ such that $1 - e = (1 - e)u(1 - a)$.
(iii) The matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is clean in $M_2(R)$.

A ring R is *weakly clean* if every $a \in R$ is weakly clean in R.