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Notes on local cohomology and duality

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ABSTRACT

We provide a formula (see Theorem 1.5) for the Matlis dual of the injective hull of $R/\mathfrak{p}$ where $\mathfrak{p}$ is a one dimensional prime ideal in a local complete Gorenstein domain $(R, \mathfrak{m})$. This is related to results of Enochs and Xu (see [4] and [5]). We prove a certain 'dual' version of the Hartshorne–Lichtenbaum vanishing (see Theorem 2.2). We prove a generalization of local duality to cohomologically complete intersection ideals I in the sense that for $I = \mathfrak{m}$ we get back the classical Local Duality Theorem. We determine the exact class of modules to which a characterization of cohomologically complete intersection from [7] generalizes naturally (see Theorem 4.4).

In this paper we prove a Matlis dual version of Hartshorne–Lichtenbaum Vanishing Theorem and generalize the Local Duality Theorem. The latter generalization is done for ideals which are cohomologically complete intersections, a notion which was introduced and studied in [7]. The generalization is such that local duality becomes the special case when the ideal I is the maximal ideal $\mathfrak{m}$ of the given local ring $(R, \mathfrak{m})$. We often use formal local cohomology, a notion which was introduced and studied by the second author in [12]. Formal local cohomology is related to Matlis duals of local cohomology modules (see [6, Sections 7.1 and 7.2] and Corollary 3.4).

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We start in Section 1 with the study of the Matlis duals of local cohomology modules $H_I^{n-1}(R)$, where $n = \dim R$. The latter is also the formal local cohomology module $\varprojlim H_{\mathfrak{m}}^1(R/I^\alpha)$ provided R is a Gorenstein ring. We describe this module as the cokernel of a certain canonical map. As a consequence we derive a formula (see Theorem 1.5) for the Matlis dual of $E_R(R/\mathfrak{p})$, where $\mathfrak{p} \in \operatorname{Spec} R$ is a 1-dimensional prime ideal. In some sense this is related to results by Enochs and Xu (see [4] and [5]).

In Section 3 we generalize the Local Duality (see Theorem 3.1). The canonical module in the classical version is replaced by the dual of $H_I^c(R)$ where I is a cohomologically complete intersection ideal of grade c (the case $I = \mathfrak{m}$ specializes to the classical local duality). See also [6, Theorem 6.4.1]. It is a little bit surprising that the d-th formal local cohomology occurs as the duality module for the duality of cohomologically complete intersections in a Gorenstein ring (see Corollary 3.4).

In Theorem 4.4 we generalize the main result [7, Theorem 3.2]. This provides a characterization of the property of 'cohomologically complete intersection' given for ideals to finitely generated modules. Some of the results of Section 4 are obtained independently by W. Mahmood (see [9]).

In the Addendum we prove a result on the behavior of direct limits in the first place of Ext-modules and the corresponding inverse limits outside (see also [10]). This could be of some independent interest. Moreover it corrects [7, Lemma 1.2 (a)].

1. On formal local cohomology

Let $(R, \mathfrak{m})$ be a local ring, let $I \subset R$ be an ideal. In the following let $\hat{R}^I$ denote the I-adic completion of R. Let $0 = \bigcap_{i=1}^r \mathfrak{q}_i$ denote a minimal primary decomposition of the zero ideal. Then we denote by $u(I)$ the intersection of those $\mathfrak{q}_i$, $i = 1, \dots, r$, such that $\dim R/(\mathfrak{p}_i + I) > 0$, where $\operatorname{Rad} \mathfrak{q}_i = \mathfrak{p}_i$, $i = 1, \dots, r$.

For the definition and basic properties of $\varprojlim H_{\mathfrak{m}}^i(R/I^\alpha)$, the so-called formal local cohomology, we refer to [12]. We denote the functor of global transform by $T(\cdot) = \varinjlim \operatorname{Hom}_R(\mathfrak{m}^\alpha, \cdot)$, in order to distinguish it from Matlis duality

$$D(M) = \operatorname{Hom}_R\bigl(M, E_R(R/\mathfrak{m})\bigr),$$

where $E_R(R/\mathfrak{m})$ is a fixed R-injective hull of $k := R/\mathfrak{m}$.

Lemma 1.1. *Let $I \subset R$ denote an arbitrary ideal. Then we get a short exact sequence*

$$0 \to \hat{R}^I/u\bigl(I\hat{R}^I\bigr) \to \varprojlim T\bigl(R/I^\alpha\bigr) \to \varprojlim H_{\mathfrak{m}}^1\bigl(R/I^\alpha\bigr) \to 0.$$

Proof. For each $\alpha \in \mathbb{N}$ use the following canonical exact sequence

$$0 \to H_{\mathfrak{m}}^0\bigl(R/I^\alpha\bigr) \to R/I^\alpha \to T\bigl(R/I^\alpha\bigr) \to H_{\mathfrak{m}}^1\bigl(R/I^\alpha\bigr) \to 0.$$

It splits up into two short exact sequences