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On the Jacobson radical of constants of derivations ☆

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ABSTRACT

Let R be a semiprimitive algebra, d its algebraic derivation and $R^d = \ker d$ the subalgebra of constants of d. It is proved that the Jacobson radical $J(R^d)$ of R^d is nilpotent. It is also shown that the following properties are equivalent: R^d is semilocal; R is semisimple Artinian; R^d is left and right Artinian.

Introduction

In [13, Theorem 2.2] Haïly et al. proved that a semiprimitive algebra containing an algebraic element, whose centralizer is semiperfect, has to be Artinian. Using this result they proved that a semiprimitive complex Banach algebra containing an element whose centralizer is algebraic has to be finite-dimensional, answering affirmatively a question

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raised in [3]. Observe that the centralizer $C_R(a)$ of an element $a \in R$ is equal to the kernel of the inner derivation induced by a. Therefore, in terms of derivations, the above result says that if the constants of a certain algebraic derivation of a semiprimitive algebra are semiperfect, then the algebra is Artinian. On the other hand it is proved in [11, Theorem 4.7] that if d is an algebraic derivation of a semiprime algebra, then R^d is left Artinian if and only if R is semisimple Artinian. Thus comparing it with [13, Theorem 2.2] we obtain that the centralizer $C_R(a)$ of an algebraic element of a semiprimitive algebra R is semiperfect if and only if $C_R(a)$ is Artinian. In particular, in this situation the Jacobson radical $J(C_R(a))$ is nilpotent. Let us remind also that if R is semiprime and d algebraic, then the prime radical $P(R^d)$ is nilpotent (see [11, Theorem 3.4]). Therefore it is reasonable to ask whether the Jacobson radical of constants of an algebraic derivation of a semiprimitive algebra has to be nilpotent. In the first main result of this paper we answer this question affirmatively and prove

Theorem A. *Suppose that either R is a semiprimitive ring with a nilpotent derivation ($d^n = 0$) or R is a semiprimitive algebra over a field K with a derivation d satisfying the identity*

$$\alpha_0 d^n + \alpha_1 d^{n+1} + \cdots + \alpha_k d^{n+k} = 0,$$

where $\alpha_0, \alpha_1, \ldots, \alpha_k \in K$ and $\alpha_0 \neq 0$. Then the Jacobson radical $J(R^d)$ is nilpotent. More precisely, $J(R^d)^\gamma = 0$, where $\gamma \leqslant 2^n - 1$.

Observe that, under assumptions of the above theorem, if R^d is semilocal, then R^d is semiperfect. In the second main result of this paper we generalize Haïly et al. result on semiperfect centralizers [13, Theorem 2.2] to the following form

Theorem B. *Suppose that either R is a semiprimitive ring with a nilpotent derivation d or R is a semiprimitive K-algebra with a K-linear algebraic derivation d. The following conditions are equivalent*:

(1) *R^d is semilocal.*
(2) *R is semisimple Artinian.*
(3) *R^d is left and right Artinian.*

We can now introduce some of the notation and terminology that will be used throughout this paper. R will denote associative rings or associative algebras over a field K. Although some of the rings and algebras in this paper may not have an identity, when examining whether an algebra is Artinian, we will assume that the algebra has an identity. By a derivation of R we mean an additive map $d\colon R \to R$ such that $d(ab) = d(a)b + ad(b)$ for $a, b \in R$. The kernel $\ker d$ we denote by R^d and call constants of d. For an element $a \in R$, ad_a stands for the inner derivation induced by a, i.e., $\mathrm{ad}_a(r) = ar - ra$ for $r \in R$.