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Constructing homogeneous Gorenstein ideals ☆

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ABSTRACT

In 1983 Kustin and Miller introduced a construction of Gorenstein ideals in local Gorenstein rings, starting from smaller such ideals. We review and modify their construction in the case of graded rings and discuss it within the framework of Gorenstein liaison theory. We determine invariants of the constructed ideal. Concerning the problem of when a given Gorenstein ideal can be obtained by the construction, we derive a necessary condition and exhibit a Gorenstein ideal that cannot be obtained using the construction.

1. Introduction

In [8] Kustin and Miller introduce a construction that produces, for given Gorenstein ideals $\mathfrak{b} \subset \mathfrak{a}$ with grades g and $g-1$, respectively, in a Gorenstein local ring R, a new Gorenstein ideal I of grade g in a larger Gorenstein ring $R[v]$. Here v is a new indeterminate. In [9] they give an interpretation for their construction via liaison theory. These

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beautiful results prompted us to review their construction for homogeneous Gorenstein ideals in a graded Gorenstein ring. Instead of introducing a new indeterminate, we use a suitable homogeneous element. The construction in [8] does not quite reveal the conditions on that homogeneous element. Therefore, we reverse the steps. We use two direct Gorenstein links to produce a new Gorenstein ideal and to describe a generating set of it. Then we adapt the original Kustin–Miller construction suitably in order to produce a graded free resolution of the new Gorenstein ideal that is often minimal. We also consider the question of when the process can be reversed, that is, when can a Gorenstein ideal be obtained using the construction.

This paper is organized as follows. In Section 2 we recall some results from liaison theory and the mapping cone procedure. In Section 3 we present a construction of homogeneous Gorenstein ideals via liaison theory. Given two homogeneous Gorenstein ideals $\mathfrak{b} \subset \mathfrak{a}$ of grades of $g-1$ and g in a graded Gorenstein ring R, by choosing an appropriate homogeneous element f in R we construct a homogeneous Gorenstein ideal $I = \mathfrak{b} + (\alpha_{g-1}^* + (-1)^g f a_g^*)$ in the original ring R. Here α_{g-1}^* and a_g^* are row vectors derived from comparing the resolutions of $\mathfrak{a}$ and $\mathfrak{b}$ and the second ideal is generated by the entries of the specified row vector (see Theorem 3.1).

Using liaison theory, one also gets a graded free resolution of I. However, this resolution is never minimal. Adapting the original Kustin–Miller construction and its proof we obtain a smaller resolution that is often minimal (see Theorem 4.1). The key is a short exact sequence, which also allows us to interpret the linkage construction in Section 2 as an elementary biliaison from $\mathfrak{a}$ on $\mathfrak{b}$. Furthermore, we obtain a necessary condition on $\mathfrak{a}$ for constructing a given Gorenstein ideal I by such a biliaison (see Corollary 4.4).

The original Kustin–Miller construction has been used to produce many interesting classes of Gorenstein ideals. In birational geometry it is known as unprojection (see, e.g., [13,14,2]). We illustrate the flexibility of our homogeneous construction by producing examples in Section 5. These include the Artinian Gorenstein ideals with socle degree two as classified by Sally [16] and the ideals of submaximal minors of a generic square matrix that are resolved by the Gulliksen–Negård complex. We also consider some Tom unprojections as studied in [2]. We conclude with an example of a homogeneous Gorenstein ideal that cannot be obtained using the construction of Theorem 3.1 with a strictly ascending biliaison.

2. Liaison and mapping cones

We frequently use ideas from liaison theory and mapping cones. We briefly recall some relevant concepts in this section.

Throughout this note R denotes a commutative Noetherian ring that is either local with maximal ideal $\mathfrak{m}$ or graded. In the latter case we assume that $R = \bigoplus_{j \geqslant 0} [R]_j$ is generated as $[R]_0$-algebra by $[R]_1$ and $[R]_0$ is a field. We denote by $\mathfrak{m}$ its maximal homogeneous ideal $\bigoplus_{j \geqslant 0} [R]_j$. If R is a graded ring, we consider only homogeneous ideals of R.