Journal of Algebra 401 (2014) 144–160

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Quantum generalized Harish-Chandra isomorphisms ☆

Polyxeni Lamprou

Weizmann Institute of Science, 76100 Rehovot, Israel

ARTICLE INFO

Article history:
Received 12 June 2013
Available online 3 January 2014
Communicated by J.T. Stafford

Keywords:
Harish-Chandra isomorphism
Quantized enveloping algebra

ABSTRACT

We give a different proof of generalized Harish-Chandra isomorphisms proven by Khoroshkin, Nazarov and Vinberg [9] and Joseph [6,7] as well as of their analogues in the quantum case.

1. Introduction

Throughout this paper, the ground field k is algebraically closed of characteristic 0.

1.1. Let $\mathfrak{g}$ be a semisimple Lie algebra, Φ its root system with respect to a fixed Cartan subalgebra $\mathfrak{h}$, π a basis in Φ and W its Weyl group. Let Λ be the weight lattice of $\mathfrak{g}$ and for any $\beta \in \Phi$, let h_β denote the corresponding coroot.

Let $(\,,\,)$ be the W-invariant symmetric bilinear form on Λ. For all $\alpha \in \pi$, set $q_\alpha := q^{(\alpha,\alpha)/2}$ and

$$[n]_\alpha = \frac{q_\alpha^n - q_\alpha^{-n}}{q_\alpha - q_\alpha^{-1}}, \qquad [n]_\alpha! = [1]_\alpha[2]_\alpha \cdots [n]_\alpha, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_\alpha = \frac{[n]_\alpha!}{[n-k]_\alpha![k]_\alpha!}.$$

1.2. Let $U_q(\mathfrak{g})$, or simply U_q, be the quantized enveloping algebra of $\mathfrak{g}$ of simply connected type. Then U_q is the algebra generated over $k(q)$ by symbols $\{E_\alpha, F_\alpha \mid \alpha \in \pi\}$ and $\{q^\lambda \mid \lambda \in \Lambda\}$ subject to the following relations.

$$\begin{aligned} &q^\lambda = 1, && \text{for } \lambda = 0. \\ &q^\lambda q^\mu = q^\mu q^\lambda = q^{\lambda+\mu}, && \text{for all } \lambda, \mu \in \Lambda. \\ &q^\lambda E_\alpha q^{-\lambda} = q^{(\alpha,\lambda)} E_\alpha = q_\alpha^{\lambda(h_\alpha)} E_\alpha, && \text{for all } \lambda \in \Lambda, \alpha \in \pi. \end{aligned}$$

☆ Work supported in part by the Israel Science Foundation Grant No. 882/10.

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2013.11.009

$$q^{\lambda}F_{\alpha}q^{-\lambda}=q^{-(\alpha,\lambda)}F_{\alpha}=q_{\alpha}^{-\lambda(h_{\alpha})}F_{\alpha}, \qquad \text{for all } \lambda\in\Lambda, \alpha\in\pi.$$

$$E_{\alpha}F_{\beta}-F_{\beta}E_{\alpha}=\delta_{\alpha,\beta}\frac{q^{\alpha}-q^{-\alpha}}{q_{\alpha}-q_{\alpha}^{-1}}, \qquad \text{for all } \alpha,\beta\in\pi.$$

$$\sum_{r=0}^{1-\beta(h_{\alpha})}(-1)^{r}\begin{bmatrix}1-\beta(h_{\alpha})\\ r\end{bmatrix}_{\alpha}E_{\alpha}^{1-\beta(h_{\alpha})-r}E_{\beta}E_{\alpha}^{r}=0, \quad \text{for all } \alpha,\beta\in\pi,\ \alpha\neq\beta.$$

$$\sum_{r=0}^{1-\beta(h_{\alpha})}(-1)^{r}\begin{bmatrix}1-\beta(h_{\alpha})\\ r\end{bmatrix}_{\alpha}F_{\alpha}^{1-\beta(h_{\alpha})-r}F_{\beta}F_{\alpha}^{r}=0, \quad \text{for all } \alpha,\beta\in\pi,\ \alpha\neq\beta.$$

The q^{λ}, $\lambda\in\Lambda$ form a multiplicative group T with $q^{-\lambda}=(q^{\lambda})^{-1}$. Denote by $U_q^0:=k(q)[T]$ the group algebra of T. Let U_q^+ (resp. U_q^-) be the subalgebra of U_q generated by the E_α (resp. the F_α), for all $\alpha\in\pi$.

1.3. The algebra U_q has a Hopf algebra structure, with $\Delta: U_q\to U_q\otimes U_q$, $\varepsilon: U_q\to k$, $S: U_q\to U_q$ the co-product, co-unit and antipode respectively defined on the generators as follows. For all $\alpha\in\pi$, $\lambda\in\Lambda$,

$$\begin{aligned}
&\Delta(E_\alpha)=E_\alpha\otimes q^{-\alpha}+1\otimes E_\alpha, && S(E_\alpha)=-E_\alpha q^{\alpha}, && \varepsilon(E_\alpha)=0,\\
&\Delta(F_\alpha)=F_\alpha\otimes 1+q^{\alpha}\otimes F_\alpha, && S(F_\alpha)=-q^{-\alpha}F_\alpha, && \varepsilon(F_\alpha)=0,\\
&\Delta\bigl(q^{\lambda}\bigr)=q^{\lambda}\otimes q^{\lambda}, && S\bigl(q^{\lambda}\bigr)=q^{-\lambda}, && \varepsilon\bigl(q^{\lambda}\bigr)=1.
\end{aligned}$$

1.4. Let $U_+^+:=\{u\in U_q^+\mid \varepsilon(u)=0\}$, $U_+^-:=\{u\in U_q^-\mid \varepsilon(u)=0\}$ denote the augmentation ideals of U_q^+, U_q^- respectively. There is a vector space decomposition of U_q as follows:

$$U_q=U_q^0\oplus\bigl(U_+^-U_q+U_qU_+^+\bigr).$$

With respect to this decomposition one defines a Harish-Chandra map $p: U_q\to U_q^0$ as projection onto the first factor. It is known that the restriction of this map on the center $Z(U_q)$ of U_q is an isomorphism onto $(U_q^0)_{ev}^{W.}$, where $(U_q^0)_{ev}$ is the subalgebra of U_q^0 generated by q^{λ}, $\lambda\in 2\Lambda$ and $W.$ denotes the translated action of the Weyl group W on the torus T, given by [4, Section 7.1.17]:

$$w.q^{\lambda}=q^{w\lambda}q^{(\rho,w\lambda-\lambda)}, \tag{1}$$

for all $\lambda\in\Lambda$, where ρ is the half sum of positive roots in Φ. Moreover, it is known that the center $Z(U_q)$ (and hence, $(U_q^0)_{ev}^{W.}$) is a polynomial algebra.

1.5. Let V be a finite-dimensional simple U_q-module of type **1** (i.e. its highest weight is q^{λ} for some $\lambda\in\Lambda$). For all $\mu\in\Lambda$ set

$$V_\mu:=\bigl\{v\in V\bigm| q^{\lambda}v=q^{(\lambda,\mu)}v,\ \text{for all } \lambda\in\Lambda\bigr\},$$

the weight space of V corresponding to μ.

We consider a left and a right action of U_q on $V\otimes U_q$ as follows. For all $x,a\in U_q$, $v\in V$ we set

$$x(v\otimes a):=\sum_i x_i v\otimes x^i a$$