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Simple rings and degree maps

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ABSTRACT

For an extension A/B of neither necessarily associative nor necessarily unital rings, we investigate the connection between simplicity of A with a property that we call A-simplicity of B. By this we mean that there is no non-trivial ideal I of B being A-invariant, that is satisfying $AI \subseteq IA$. We show that A-simplicity of B is a necessary condition for simplicity of A for a large class of ring extensions when B is a direct summand of A. To obtain sufficient conditions for simplicity of A, we introduce the concept of a degree map for A/B. By this we mean a map d from A to the set of non-negative integers satisfying the following two conditions: (d1) if $a \in A$, then $d(a) = 0$ if and only if $a = 0$; (d2) there is a subset X of B generating B as a ring such that for each non-zero ideal I of A and each non-zero $a \in I$ there is a non-zero $a' \in I$ with $d(a') \leqslant d(a)$ and $d(a'b - ba') < d(a)$ for all $b \in X$. We show that if the centralizer C of B in A is an A-simple ring, every intersection of C with an ideal of A is A-invariant, $ACA = A$ and there is a degree map for A/B, then A is simple. We apply these results to various types of graded and filtered rings, such as skew group rings, Ore extensions and Cayley–Dickson doublings.

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1. Introduction

Let A/B be a ring extension. By this we mean that A and B are rings that are neither necessarily associative nor necessarily unital with B contained in A. For general ring extensions, simplicity of A is, of course, neither a necessary nor a sufficient condition for simplicity of B. However, in many cases where A is graded or filtered, and B sits in A as a direct summand, simplicity of A is connected to weaker simplicity conditions for B. In particular, this often holds if there is an action on the ring B induced by the ring structure on A. The aim of this article is to investigate both necessary (see Theorem 1.3) and sufficient (see Theorem 1.4) conditions on B for simplicity of A. The impetus for our approach is threefold.

The first part of our motivation comes from group graded ring theory and topological dynamics. A lot of attention has been given the connection between properties of topological dynamical systems (X, s) and the ideal structure of the corresponding C^*-algebras $C^*(X, s)$ (see e.g. [3,17,20,21,23]). In particular, Power [17] has shown that if X is a compact Hausdorff space of infinite cardinality, then $C^*(X, s)$ is simple if and only if (X, s) is minimal. Inspired by this result, Öinert [11,16] has shown that there is an analogous result for skew group algebras defined by general group topological dynamical systems. In fact Öinert [11,16] shows this by first establishing the following result for general skew group rings (for more details concerning topological dynamics, see Section 3.6).

Theorem 1.1. *(See Öinert [11,16].) Suppose that B is an associative, unital ring and $A = B \rtimes^\sigma G$ is a skew group ring.* (a) *If B is commutative, then A is simple if and only if B is G-simple and B is a maximal commutative subring of A*; (b) *If G is abelian, then A is simple if and only if B is G-simple and $Z(A)$ is a field.*

The second part of our motivation comes from the filtered class of rings called Ore extensions $A = B[x; \sigma, \delta]$. A lot of work has been devoted to the study of the connection between the ideal structure of A and B in this situation (see e.g. [2,6–8,15]). In [15] Öinert, Richter and Silvestrov show that there are simplicity results for differential polynomial rings that are almost completely analogous to the skew group ring situation (for more details, see Section 3.7).

Theorem 1.2. *(See Öinert, Richter and Silvestrov [15].) Suppose that B is an associative and unital ring and $A = B[x; \delta]$ is a differential polynomial ring.* (a) *If B is a δ-simple and maximal commutative subring of the differential polynomial ring A, then A is simple*; (b) *The ring A is simple if and only if B is δ-simple and $Z(A)$ is a field.*

The third part of our motivation comes from non-associative ring theory. Namely, starting with a field B_0 of characteristic different from two, and repeatedly applying the Cayley–Dickson doubling procedure (see Section 3.3 and Section 3.4), we get an infinite chain B_i, for $i \geqslant 0$, of B_0-algebras. Although none of the rings B_i, for $i \geqslant 3$, is associative, it is well known that they are all simple (see e.g. [5]).