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Functional identities in one variable ☆

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ABSTRACT

Let A be a centrally closed prime algebra over a characteristic 0 field k, and let $q : A \to A$ be the trace of a d-linear map (i.e., $q(x) = M(x, \dots, x)$ where $M : A^d \to A$ is a d-linear map). If $[q(x), x] = 0$ for every $x \in A$, then q is of the form $q(x) = \sum_{i=0}^{d} \mu_i(x)x^i$ where each μ_i is the trace of a $(d-i)$-linear map from A into k. For infinite dimensional algebras and algebras of dimension $> d^2$ this was proved by Lee, Lin, Wang, and Wong in 1997. In this paper we cover the remaining case where the dimension is $\leqslant d^2$. Using this result we are able to handle general functional identities in one variable on A; more specifically, we describe the traces of d-linear maps $q_i : A \to A$ that satisfy $\sum_{i=0}^{m} x^i q_i(x) x^{m-i} \in k$ for every $x \in A$.

1. Introduction

Let R be a ring. A map $q : R \to R$ is said to be *commuting* if $[q(x), x] = q(x)x - xq(x) = 0$ for all $x \in R$. The study of commuting maps has a long and rich history, starting with Posner's 1957 theorem stating that there are no nonzero commuting derivations on noncommutative prime rings [16]. The reader is referred to the survey

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article [8] for the general theory of commuting maps. We will be concerned with commuting traces of multilinear maps on prime algebras. A map q between additive groups (resp. vector spaces) A and B is called the trace of a d-additive (resp. d-linear) map if there exists a d-additive (resp. d-linear) map $M : A^d \to B$ such that $q(x) = M(x, \dots, x)$ for all $x \in A$ (for $d = 0$ this should be understood as that q is a constant). If R is a prime ring (resp. algebra) and $q : R \to R$ is a commuting trace of a d-additive map, is then q necessarily of a *standard form*, meaning that there exist traces of $(d-i)$-additive (resp. $(d-i)$-linear) maps μ_i from R into the extended centroid C of R such that $q(x) = \sum_{i=0}^{d} \mu_i(x)x^i$ for all $x \in R$? This question was initiated in 1993 by the first author who obtained an affirmative answer for $d = 1$ [5], and also for $d = 2$ [6] provided that $\operatorname{char}(R) \neq 2$ and R does not satisfy S_4, the standard polynomial identity of degree 4. The $d = 3$ case was treated, in a slightly different context, by Beidar, Martindale, and Mikhalev [3]. These three results have turned out to be applicable to various problems, particularly in the Lie algebra theory, and have played a crucial role in the development of the theory of functional identities [9]. The next step was made in 1997 by Lee, Lin, Wang, and Wong [15], who answered the above question in affirmative for a general d, but under the assumption that $\operatorname{char}(R) = 0$ or $\operatorname{char}(R) > d$ and R does not satisfy the standard polynomial identity S_{2d}. The reason for the exclusion of rings satisfying polynomial identities (of low degrees) is the method of the proof; it is, therefore, natural to ask whether, by a necessarily different method, one can get rid of this assumption. For $d = 2$ this has turned out to be the case [10]. To the best of our knowledge, the question is still open for a general d (cf. [8, p. 377] and [9, p. 130]). The first main goal of this paper is to give an affirmative answer for centrally closed prime algebras and traces of d-linear maps (Theorem 3.3). In this setting the problem can be reduced to the case where the algebra in question is a full matrix algebra. The essence of our approach is interpreting the reduced problem in the algebra of generic matrices, which enables to use the tools of commutative algebra.

In the second part of the paper, in Sections 4–7, we will deal with considerably more general functional identities. However, the main issue will be the reduction to the commuting trace case. Roughly speaking, the general theory of functional identities deals with expressions of the form

$$\sum x_{i_1} \cdots x_{i_p} F_t(x_{j_1}, \dots, x_{j_q}) x_{k_1} \cdots x_{k_r},$$

which are assumed to be either 0 or central for all elements $x_{i_1}, \dots, x_{k_r}$; here, F_t are arbitrary functions, and the goal is to describe their form [9]. One usually assumes that the ring (algebra) in question does not satisfy nontrivial polynomial identities (of low degrees), which makes the problem essentially easier. The second main goal of this paper is to study functional identities in one variable in general centrally closed prime algebras. That is, we will consider the situation where the summation $\sum_{i=0}^{m} x^i q_i(x) x^{m-i}$ is always zero or central; here, q_i will be assumed to be the traces of multilinear maps (so that after the linearization one gets an expression as above). The main novelty is that we will