Journal of Algebra 401 (2014) 245–257

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On nilpotent and solvable Lie algebras of derivations

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ARTICLE INFO

Article history:
Received 15 June 2013
Available online 8 January 2014
Communicated by Volodymyr Mazorchuk

MSC:
primary 17B66
secondary 17B05, 13N15

Keywords:
Lie algebra
Vector field
Solvable algebra
Derivation
Commutative ring

ABSTRACT

Let K be a field and A be a commutative associative K-algebra which is an integral domain. The Lie algebra $\operatorname{Der}_K A$ of all K-derivations of A is an A-module in a natural way, and if R is the quotient field of A then $R \operatorname{Der}_K A$ is a vector space over R. It is proved that if L is a nilpotent subalgebra of $R \operatorname{Der}_K A$ of rank k over R (i.e. such that $\dim_R RL = k$), then the derived length of L is at most k and L is finite dimensional over its field of constants. In case of solvable Lie algebras over a field of characteristic zero their derived length does not exceed $2k$. Nilpotent and solvable Lie algebras of rank 1 and 2 (over R) from the Lie algebra $R \operatorname{Der}_K A$ are characterized. As a consequence we obtain the same estimations for nilpotent and solvable Lie algebras of vector fields with polynomial, rational, or formal coefficients. Analogously, if X is an irreducible affine variety of dimension n over an algebraically closed field K of characteristic zero and A_X is its coordinate ring, then all nilpotent (solvable) subalgebras of $\operatorname{Der}_K A_X$ have derived length at most n ($2n$ respectively).

Introduction

Let $\mathbb{K}$ be a field and A be an associative commutative $\mathbb{K}$-algebra with unity, without zero divisors, i.e. an integral domain. The set $\operatorname{Der}_{\mathbb{K}} A$ of all $\mathbb{K}$-derivations of A, i.e.

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0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2013.11.021

$\mathbb{K}$-linear operators D on A satisfying the Leibniz rule: $D(ab) = D(a)b + aD(b)$ for all $a, b \in A$ is a Lie algebra over $\mathbb{K}$ and an A-module in a natural way: given $a \in A$, $D \in \operatorname{Der}_{\mathbb{K}} A$, the derivation aD sends any element $x \in A$ to $a \cdot D(x)$. The structure of the Lie algebra $\operatorname{Der}_{\mathbb{K}} A$ is of great interest because, in geometric terms, derivations can be considered as vector fields on geometric objects. For example, in case $\mathbb{K} = \mathbb{C}$ and $A = \mathbb{C}[x_1, \dots, x_n]$, the polynomial ring, any $D \in \operatorname{Der}_{\mathbb{K}} A$ is of the form

$$D = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}, \quad f_i \in \mathbb{C}[x_1, \dots, x_n],$$

i.e. D is a vector field on $\mathbb{C}^n$ with polynomial coefficients. Lie algebras of vector fields with polynomial, formal power series, or analytical coefficients were studied intensively by many authors (see, for example, [7,1–4,11,12]).

In general case, when A is an integral domain, subalgebras L of $\operatorname{Der}_{\mathbb{K}} A$ such that L are submodules of the A-module $\operatorname{Der}_{\mathbb{K}} A$ were studied in [6] (see also [10,13]), and sufficient conditions were given for L to be simple. In this paper, we study subalgebras of the Lie algebra $\operatorname{Der}_{\mathbb{K}} A$ at the other extreme: nilpotent and solvable, under the condition that they are of finite rank over A. Recall that if R is the quotient field of A, then the rank $\operatorname{rk}_R L$ is defined as $\operatorname{rk}_R L = \dim_R RL$. Any subalgebra L of the Lie algebra $\operatorname{Der}_{\mathbb{K}} A$ determines uniquely the field $F = F(L)$ of constants consisting of all $r \in R$ such that $D(r) = 0$ for all $D \in L$. The vector space FL over the field F is actually a Lie algebra over F (note that RL being a Lie algebra over $\mathbb{K}$ is not in general a Lie algebra over R). The main results of the paper: if L is a nilpotent subalgebra of the Lie algebra $R \operatorname{Der}_{\mathbb{K}} A$ with $\operatorname{rk}_R L = k$, then the derived length of L is at most k and the Lie algebra FL is finite dimensional over F (Theorem 1). In case when L is solvable, $\operatorname{rk}_R L = k$ and $\operatorname{char} \mathbb{K} = 0$, the derived length does not exceed $2k$ (Theorem 2). If $\dim_{\mathbb{K}} L < \infty$, then the last estimation can be improved to $k + 1$.

If we consider the important case $\mathbb{K} = \mathbb{C}$ and $A = \mathbb{C}[[x_1, \dots, x_n]]$, the ring of formal power series, we get that nilpotent (solvable) subalgebras of the Lie algebra $\operatorname{Der}_{\mathbb{K}} A$ of rank k over R have derived length $\leqslant k$ ($\leqslant 2k$, respectively). Note that in this particular case it was proved in [9] that all nilpotent subalgebras have derived length at most n and solvable at most $2n$ (see Corollary 3).

One can apply obtained results for vector fields on an affine variety X and obtain analogous bounds for the derived length of nilpotent and solvable subalgebras of the Lie algebra $\operatorname{Der}_{\mathbb{K}} A_X$ where A_X is the coordinate ring of X (see Corollary 4).

We also give a rough characterization of nilpotent and solvable subalgebras of rank 1 and 2 over R from the Lie algebra $R \operatorname{Der}_{\mathbb{K}} A$ (over their fields of constants). Such a characterization can be applied to study finite dimensional Lie algebras of smooth vector fields in three variables (the case of one and two variables was studied in [7,3,4]). Using the same approach we gave in [8] a description of finite dimensional subalgebras of $W(A)$ in case $A = \mathbb{K}(x, y)$, the field of rational functions.

We use standard notations, the ground field $\mathbb{K}$ is arbitrary unless otherwise stated. The quotient field of the integral domain A under consideration will be denoted by R.