Journal of Algebra 400 (2014) 8–16

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

A generalization of the Burnside basis theorem

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ARTICLE INFO

Article history:
Received 10 February 2013
Available online 20 December 2013
Communicated by Ronald Solomon

Keywords:
Finite groups
Basis
Minimal generating sets
Burnside basis theorem
Matroids

ABSTRACT

A $\mathcal{B}$-group is a group such that all its minimal generating sets (with respect to inclusion) have the same size. We prove that the class of finite $\mathcal{B}$-groups is closed under taking quotients and that every finite $\mathcal{B}$-group is solvable. Via a complete classification of Frattini-free finite $\mathcal{B}$-groups we obtain a general structure theorem for finite $\mathcal{B}$-groups. Applications include new proofs for the characterization of finite matroid groups.

1. Introduction

Let G be a finite group. A generating set X of G is said to be *minimal* if no proper subset of X generates G. We denote by $d(G)$ the minimal number of generators of G, i.e., the smallest size of a minimal generating set of G, and we write $m(G)$ for the largest size of a minimal generating set of G.

Whereas the invariant $d(G)$ has been well studied for many groups G, its counterpart $m(G)$ has not received a similar degree of attention. First steps toward investigating the latter have been taken in the context of permutation groups. For instance, in [10] Whiston proved that $m(\mathrm{Sym}(n)) = n - 1$ for the finite symmetric group of degree n. Furthermore, Cameron and Cara gave in [2] a complete description of the maximal independent generating sets of $\mathrm{Sym}(n)$; these are precisely the minimal generating sets of maximal size. Clearly, Whiston's result implies that

$$m\big(\mathrm{Sym}(n)\big) - d\big(\mathrm{Sym}(n)\big) \to \infty \quad \text{as } n \to \infty.$$

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0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2013.11.005

This suggests a natural 'classification problem': given a non-negative integer c, characterize all finite groups G such that $m(G) - d(G) \leqslant c$. Since the Frattini subgroup $\Phi(G)$ consists of all 'non-generators' of G, we have $d(G) = d(G/\Phi(G))$ and $m(G) = m(G/\Phi(G))$. Hence one may initially focus on groups G which are *Frattini-free*, i.e., for which $\Phi(G) = 1$.

In the present article we solve the above stated problem for $c = 0$. We say that the group G has property $\mathcal{B}$, or the *weak basis property*, if all its minimal generating sets have the same size, equivalently if $m(G) = d(G)$. Groups with property $\mathcal{B}$ are called $\mathcal{B}$-groups for short. A group is said to have the *basis property* if all its subgroups have property $\mathcal{B}$. The Burnside basis theorem states that all finite p-groups are $\mathcal{B}$-groups and, consequently, have the basis property.

Groups with the basis property as well as variants, such as matroid groups, have been considered by a number of authors; e.g., see [5] and the references therein. Indeed, McDougall-Bagnall and Quick initiated in [5] the systematic study of finite $\mathcal{B}$-groups and used this to classify groups with the basis property. Recently a mistake was discovered in their main theorem; see Appendix A. The classification of groups with the basis property is therefore again an open problem. Regarding finite $\mathcal{B}$-groups, McDougall-Bagnall and Quick raised the following fundamental questions. Is it true that property $\mathcal{B}$ is inherited by quotient groups? Is it possibly true that every finite $\mathcal{B}$-group is solvable? We answer both questions positively.

Proposition 1.1. *Every quotient of a finite $\mathcal{B}$-group is again a $\mathcal{B}$-group.*

Theorem 1.2. *Every finite $\mathcal{B}$-group is solvable.*

Remark 1.3. The results of Saxl and Whiston in [7] show that for projective special linear groups $G = \mathrm{PSL}_2(p^r)$ the difference $m(G) - d(G)$ depends on the number of prime divisors of r. In particular, $m(G) - d(G) = 1$ for all $G = \mathrm{PSL}_2(p)$ with p not congruent to ± 1 modulo 8 or 10. Therefore generalizing Theorem 1.2 to groups G with $m(G) - d(G) \leqslant 1$ is an interesting open problem.

From Proposition 1.1 and Theorem 1.2 we obtain a characterization of finite $\mathcal{B}$-groups, based on a complete classification of Frattini-free finite $\mathcal{B}$-groups. For any prime p we denote by $\mathbb{F}_p$ the field with p elements.

Theorem 1.4. *Let G be a finite group. Then G is a Frattini-free $\mathcal{B}$-group if and only if one of the following holds*:

(1) *G is an elementary abelian p-group for some prime p*;
(2) *$G = P \rtimes Q$, where P is an elementary abelian p-group and Q is a non-trivial cyclic q-group, for distinct primes $p \neq q$, such that Q acts faithfully on P and the $\mathbb{F}_p[Q]$-module P is isotypical, i.e., a direct sum of $m(G) - 1$ isomorphic copies of one simple module.*

Remark 1.5. This means that there are no Frattini-free finite $\mathcal{B}$-groups beyond the examples constructed in [5, §3]. Indeed, the groups listed in (2) of Theorem 1.4 can be concretely realized as 'semidirect products via multiplication in finite fields of characteristic p': the simple module in question is of the form $\mathbb{F}_p(\zeta)$, the additive group of a finite field generated by a q^kth root of unity ζ over $\mathbb{F}_p$, with a generator z of Q acting on $\mathbb{F}_p(\zeta)$ as multiplication by ζ.

Using the explicit description of Frattini-free finite $\mathcal{B}$-groups, we determine the automorphism groups of such groups; see Theorem 4.1. From McDougall-Bagnall and Quick's results in [5] we obtain a characterization of finite $\mathcal{B}$-groups.

Theorem 1.6. *Let G be a finite group. Then G is a $\mathcal{B}$-group if and only if one of the following holds*:

(1) *G is a p-group for some prime p*;
(2) *$G = P \rtimes Q$, where P is a p-group and Q is a cyclic q-group, for distinct primes $p \neq q$, such that $C_Q(P) \neq Q$ and the $\mathbb{F}_p[Q/C_Q(P)]$-module $P/\Phi(P)$ is isotypical, i.e., a direct sum of isomorphic copies of one simple module.*

Moreover, in case (2) *one has $\Phi(G) = \Phi(P) \times C_Q(P)$.*