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Discriminants of symplectic graded involutions on graded simple algebras

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ABSTRACT

In this article, we define and study the discriminant of symplectic graded involutions on non-inertially split graded simple algebras with simple 0-component. In particular, we show that if F is a graded field of characteristic different from 2, D is a graded central division algebra over F with $\exp(D) = 2$ and $|\ker(\theta_D)| > 4$ (see the preliminaries below), $A = M_n(D)$, and σ is a graded involution of symplectic type on A, then there is only a finite number of values for the discriminants $\Delta_\sigma(\tau)$, where τ describes all graded involutions of symplectic type on A (see Proposition 2.11). Consequently, for any graded central simple algebra C over F with C_0 simple non-split, $\exp(C) = 2$, $|\ker(\theta_C)| > 4$ and $\frac{\deg(C)}{\operatorname{ind}(C)}$ even, we have $\Delta_\sigma(\tau) = 0$ for any graded involutions of symplectic type σ and τ on C (see Corollary 2.12). We prove also that if E is a Henselian valued field with residue characteristic different from 2, D is a central division algebra of exponent 2 over E with $|\ker(\theta_D)| > 4$, and $B = M_n(D)$ with n even, then for any symplectic involutions σ, τ on B, preserving a tame gauge defined on B, we have $\Delta_\sigma(\tau) = 0$ (see Corollary 3.5).

Let E be a field and A be a central simple algebra over E. We recall that if there exists a symplectic involution σ on A, then $\deg(A)$ is even, say $\deg(A) = 2m$. The corresponding pfaffian reduced norm is defined to be the homogeneous polynomial function of degree m:

$$\mathrm{Nrp}_\sigma : \mathrm{Sym}(A, \sigma) \to E$$

uniquely determined by the following conditions:

$\mathrm{Nrp}_\sigma(1) = 1$ and $\mathrm{Nrp}_\sigma(x)^2 = \mathrm{Nrd}_A(x)$ for all $x \in \mathrm{Sym}(A, \sigma)$, where $\mathrm{Sym}(A, \sigma)$ is the E-space of symmetric elements of A under σ.

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Let $\mathrm{G_m}$ be the multiplicative group scheme. The Kummer exact sequence:

$$1 \to \mu_2 \to \mathrm{G_m} \xrightarrow{(\,)^2} \mathrm{G_m} \to 1$$

allows us to identify the cohomology group $\mathrm{H}^1(E, \mu_2)$ with the quotient group E^*/E^{*2} and the cohomology group $\mathrm{H}^2(E, \mu_2)$ with the 2-torsion subgroup ${}_2\mathrm{Br}(E)$ of the Brauer group of E (see [11, p. 413]). For $a \in E^*$, let $(a)_2 \in \mathrm{H}^1(E, \mu_2)$ be the cohomology class associated to aE^{*2}, and let $[A] \in \mathrm{H}^2(E, \mu_2)$ be the cohomology class associated to the Brauer class of A. A map Δ_σ : $\mathrm{Sym}(A, \sigma)^* \to \mathrm{H}^3(E, \mu_2)$ is then defined by $\Delta_\sigma(a) = (\mathrm{Nrp}_\sigma(a))_2 \cup [A]$, where $\mathrm{Sym}(A, \sigma)^*$ is the set of invertible elements of $\mathrm{Sym}(A, \sigma)$, and where $\cup$ is the cup-product. If $\deg(A) \equiv 0 \bmod 4$ and τ is another symplectic involution on A, then the discriminant $\Delta_\sigma(\tau)$ is defined to be $\Delta_\sigma(a)$, where a is an arbitrary element of $\mathrm{Sym}(A, \sigma)^*$ such that $\tau = \mathrm{Int}(a) \circ \sigma$. This discriminant depends only on the conjugacy classes of σ and τ (see [3, Proposition 1(a)]). It is then an invariant of symplectic involutions on A.

Berhuy, Monsurrò and Tignol defined this invariant in [3] on the basis of Rost's cohomological invariant of degree 3 for torsors under symplectic groups. They established in [3, Theorem 4, Corollary 5 and Proposition 6] relationships between this discriminant and trace forms. They also showed that for a central simple algebra A of degree 8 and index 4 with a symplectic involution σ, the triviality of $\Delta(\sigma) := \Delta_\alpha(\sigma)$, where α is a hyperbolic involution on A, is equivalent to a special decomposition of σ on quaternion subalgebras of A (see [3, Theorem 8]).

Garibaldi, Parimala and Tignol showed in [7] that this (relative) discriminant leads to a unique absolute invariant in the sense of [6] of all symplectic involutions on simple algebras of a fixed degree n divisible by 8, or equivalently a cohomological invariant of the split adjoint group PGSp_{2m} of type C_m, for m divisible by 4 since the cohomological set $\mathrm{H}^1(K, \mathrm{PGSp}_{2m})$ classifies central simple K-algebras of degree $2m$ endowed with a symplectic involution for each extension K of the base field E (see [7, Theorem A]). In particular they showed a decomposition criterion of symplectic involutions on central simple algebras of degree 8 in terms of the triviality of this absolute invariant.

Recently, Tignol and Wadsworth defined tame gauges on central simple algebras over valued fields, which in the case of a Henselian valued field E, extend tame valuations on central division algebras over E. They established then correspondences relating graded simple algebras to simple algebras with tame gauges and showed that the graded techniques simplify significantly some previous arguments used to prove results on central simple algebras (see [17]). Using these correspondences, other works showed that graded involutions on graded simple algebras can serve as a powerful tool to study properties of involutions on central simple algebras when these algebras are endowed with tame gauges (see for example [12] and [18]). In this context, we showed in [13] that the results of Chacron et al. [4] concerning the computation of the discriminant of orthogonal involutions on central division algebras over Henselian fields can be extended to some orthogonal involutions on central simple algebras with tame gauges and simple residue algebras over a large family of Henselian valued fields.

In the present article, we aim to extend this study concerning the discriminant invariant to cover the case where the graded involution is symplectic. We focus essentially on symplectic graded involutions on non-inertially split graded simple algebras with simple 0-components. More precisely, we show that if F is a graded field of characteristic different from 2, D is a graded central division algebra over F with $\exp(D) = 2$ and $|\ker(\theta_D)| > 4$, $A = M_n(D)$, and σ is a graded involution of symplectic type on A, then there is only a finite number of values for the discriminants $\Delta_\sigma(\tau)$, where τ describes all graded involutions of symplectic type on A (see Proposition 2.11). Consequently, for any graded central simple algebra C over F with C_0 simple non-split, $\exp(C) = 2$, $|\ker(\theta_C)| > 4$ and $\frac{\deg(C)}{\mathrm{ind}(C)}$ even, we have $\Delta_\sigma(\tau) = 0$ for any graded involutions of symplectic type σ and τ on C (see Corollary 2.12). Analogously, and as application of some arguments used for the graded case, we prove in Section 3 that if E is a Henselian valued field with residue characteristic different from 2, D is a central division algebra of exponent 2 over E such that $8 \leqslant |\ker(\theta_D)|^{\frac{1}{2}}$, and σ, τ are symplectic involutions on D, then $\Delta_\sigma(\tau) = 0$ (see Corollary 3.3). We show also that if D is a central division algebra over E with $|\ker(\theta_D)| > 4$, $B = M_n(D)$ with n even, β is the tame gauge defined on B by