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Poisson spectra in polynomial algebras

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ABSTRACT

A significant class of Poisson brackets on the polynomial algebra $\mathbb{C}[x_1,\dots,x_n]$ is studied and, for this class of Poisson brackets, the Poisson prime ideals, Poisson primitive ideals and symplectic cores are determined. Moreover it is established that these Poisson algebras satisfy the Poisson Dixmier–Moeglin equivalence.

1. Introduction

To understand a Poisson bracket on the polynomial algebra $\mathbb{C}[x_1,\dots,x_n]$, one should identify the Poisson prime ideals, which correspond to the varieties in $\mathbb{C}^n$ that inherit the Poisson structure, and the *symplectic cores* [2] which are the algebraic analogues of symplectic leaves. For each point $p \in \mathbb{C}^n$, with corresponding maximal ideal M_p, there is a unique largest Poisson ideal $\mathcal{P}(M_p)$ contained in M_p. The ideal $\mathcal{P}(M_p)$, which is necessarily Poisson prime, is said to be *Poisson primitive* and is called the *Poisson core* of M_p. Two points p and q are in the same symplectic core when $\mathcal{P}(M_p) = \mathcal{P}(M_q)$. See [5, Sections 6, 7] for a full discussion of symplectic cores and their relationship with symplectic leaves.

In [10], the authors analyzed Poisson brackets on the polynomial algebra $\mathbb{C}[x,y,z]$ in three indeterminates x,y,z, including a class of Poisson brackets determined by Jacobians. In particular, for an

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arbitrary rational function $s/t \in \mathbb{C}(x, y, z)$, they analyzed the prime and primitive Poisson ideals for the Poisson bracket such that, for $f, g \in \mathbb{C}[x, y, z]$,

$$\{f, g\} = t^2 \operatorname{Jac}(f, g, s/t), \tag{1.1}$$

where Jac denotes the Jacobian determinant.

The main purpose of this paper is to generalize the results in [10] to the general polynomial algebra $A := \mathbb{C}[x_1, x_2, \dots, x_n]$, $n \geqslant 3$, equipped with a Poisson bracket which is determined by $n-2$ rational functions and which generalizes (1.1). As in [10], the results will be illustrated using particular examples.

Fix $s_1, t_1, \dots, s_{n-2}, t_{n-2} \in A$ such that s_i and $t_i \neq 0$ are coprime for each $i = 1, 2, \dots, n-2$. In Section 2 it is shown that there is a Poisson bracket on the quotient field B of A such that, for all $f, g \in B$,

$$\{f, g\} = (t_1 \dots t_{n-2})^2 \operatorname{Jac}(f, g, s_1/t_1, s_2/t_2, \dots, s_{n-2}/t_{n-2}).$$

The purpose of the factor $(t_1 \dots t_{n-2})^2$ is to ensure that this restricts to a Poisson bracket on A.

The Poisson prime ideals of A for the above bracket are determined in Section 3, where Definition 3.2 uses the terminology *residually null*, respectively *proper*, for Poisson prime ideals P where the induced Poisson bracket on A/P is zero, respectively non-zero. The residually null Poisson prime ideals of A form a Zariski closed set of the prime spectrum of A and can often be found explicitly using elementary commutative algebra. We shall determine the proper Poisson prime ideals of A in terms of a finite set of localizations A_γ of A, each of which has a subalgebra C_γ that is a polynomial ring in $n-2$ variables and is contained in the Poisson centre of A_γ. As the Poisson bracket on C_γ is trivial, any prime ideal Q of C_γ is Poisson. Although QA_γ need not be prime, it is a Poisson ideal and the finitely many minimal prime ideals of A_γ over QA_γ are Poisson prime ideals of A_γ. Taking the intersection of each of these with A, we obtain finitely many Poisson prime ideals of A. The main result is that every proper Poisson prime ideal P of A_γ occurs in this way with $Q = PA_\gamma \cap C_\gamma$. The passage between Poisson prime ideals of A_γ and those of A can then be handled by standard localization techniques. This will be illustrated using examples with $n = 4$ in which case the algebras C_γ are polynomial algebras in two indeterminates. The main example is the Poisson bracket associated with 2×2 quantum matrices with which the reader may be familiar. We also consider actions on A, as Poisson automorphisms, of subgroups of the multiplicative group $(\mathbb{C}^*)^n$.

In Section 4, we determine the Poisson primitive ideals and symplectic cores of A and show that A satisfies the Poisson Dixmier–Moeglin equivalence discussed in [12, 2.4] and [6]. Here, as indeed is the case with the Poisson prime ideals, the varieties determined by $n-2$ polynomials of the form $\lambda_i s_i - \mu_i t_i$, $i = 1, 2, \dots, n-2$, where $(\lambda_i, \mu_i) \in \mathbb{C}^2 \setminus \{(0,0)\}$ for all i, play an important role.

2. Poisson brackets

Definition 2.1. A *Poisson algebra* is a $\mathbb{C}$-algebra A with a Poisson bracket, that is a bilinear product $\{-,-\} : A \times A \to A$ such that A is a Lie algebra under $\{-,-\}$ and, for all $a \in A$, the *hamiltonian* $\operatorname{ham}(a) := \{a, -\}$ is a $\mathbb{C}$-derivation of A.

Notation 2.2. Let A denote the polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$ in n indeterminates and let B denote the quotient field $\mathbb{C}(x_1, \dots, x_n)$ of A. For $1 \leqslant i \leqslant n$, let ∂_i be the derivation $\frac{\partial}{\partial x_i}$ of B. For $b_1, \dots, b_n \in B$, let $\operatorname{Jac}_M(b_1, \dots, b_n)$ denote the Jacobian matrix $(\partial_j(b_i))$ and let $\operatorname{Jac}(b_1, \dots, b_n)$ denote the Jacobian determinant $|\operatorname{Jac}_M(b_1, \dots, b_n)|$. Thus the ith row of $\operatorname{Jac}_M(b_1, \dots, b_n)$ is $\nabla(b_i)$ where $\nabla = (\partial_1, \partial_2, \dots, \partial_n)$ is the gradient.

Let $a, f_1, f_2, f_3, \dots, f_{n-2} \in B$ and, for $f, g \in B$, let

$$\{f, g\} = a \operatorname{Jac}(f, g, f_1, f_2, \dots, f_{n-2}). \tag{2.1}$$