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Finding inverse systems from coordinates

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ABSTRACT

Let I be a homogeneous ideal in $R = \mathbb{K}[x_0, \dots, x_n]$, such that R/I is an Artinian Gorenstein ring. A famous theorem of Macaulay says that in this instance I is the ideal of polynomial differential operators with constant coefficients that cancel the same homogeneous polynomial F. A major question related to this result is to be able to describe F in terms of the ideal I. In this note we give a partial answer to this question, by analyzing the case when I is the Artinian reduction of the ideal of a reduced (arithmetically) Gorenstein zero-dimensional scheme $\Gamma \subset \mathbb{P}^n$. We obtain F from the coordinates of the points of Γ.

1. Introduction

Let $\mathbb{K}$ be a field of characteristic zero, and let $R = \mathbb{K}[x_0, \dots, x_n]$ be the ring of homogeneous polynomials with coefficients in $\mathbb{K}$. Let $I \subset R$ be a homogeneous ideal.

The ring R/I is called *Artinian* if R/I is a finite dimensional vector space over $\mathbb{K}$. Equivalently, there exists a positive integer $d > 0$ such that $(R/I)_d = 0$ (i.e., every homogeneous polynomial of degree d is an element of I).

An Artinian ring R/I is called *Gorenstein* if the *socle*

$$Soc(R/I) := \{ a \in R/I \mid \langle x_0, \dots, x_n \rangle a = 0 \}$$

is a 1-dimensional $\mathbb{K}$-vector space. If $s + 1$ is the least integer such that $(R/I)_{s+1} = 0$ and if R/I is Gorenstein, then $Soc(R/I) = (R/I)_s$ and therefore, $\dim(R/I)_s = 1$. In this instance s is called the *socle degree* of R/I.

An *arithmetically Gorenstein* scheme means a projective scheme whose coordinate ring localized at any of its prime ideals is a local Gorenstein ring (i.e., it is a local ring that is Cohen–Macaulay and

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the canonical module is free of rank 1). In terms of the graded minimal free resolution, if $X \subset \mathbb{P}^n$ is a d-dimensional scheme with defining ideal I_X, then X is arithmetically Gorenstein if and only if R/I_X has the graded minimal free resolution as an R-module:

$$0 \to F_k \to \cdots \to F_1 \to R \to R/I_X \to 0,$$

where $k = n - d$ and $F_k \simeq R(-\alpha)$. $\alpha - k$ will be called also the *socle degree* of X, for obvious reasons, and coincides with the Castelnuovo–Mumford regularity $reg(R/I_X)$ of X.

A famous theorem of Macaulay [11], known as *Macaulay's Inverse System Theorem*, states that the Artinian ring R/I is Gorenstein if and only if I is the ideal of a system of homogeneous polynomial differential operators with constant coefficients having a unique solution. More precisely, let $S = \mathbb{K}[y_0, \dots, y_n]$ be the homogeneous polynomial ring with coefficients in $\mathbb{K}$ and variables $y_0, \dots, y_n$. R acts on S by

$$x_0^{i_0} \cdots x_n^{i_n} \circ y_0^{j_0} \cdots y_n^{j_n} = \frac{\partial^{i_0+\cdots+i_n}}{\partial y_0^{i_0} \cdots \partial y_n^{i_n}} \left(y_0^{j_0} \cdots y_n^{j_n}\right),$$

extended by linearity. Then R/I is Artinian Gorenstein if and only if $I = Ann(F) := \{f \in R \mid f \circ F = 0\}$, for some $F \in S$ (we are going to denote the elements in S with capital letters). The best surveys on applications of inverse systems and also very good introductions to this subject are [6] and [9], and the citations therein.

The polynomial F is roughly what is known as the inverse system of the Artinian Gorenstein ideal I. In general, for any ideal I, *the inverse system* of I, is by definition: $I^{-1} := \bigoplus_j (I^{-1})_j$, where

$$\left(I^{-1}\right)_j := \{G \in S_j \mid f \circ G = 0, \text{ for any } f \in I_j\}.$$

If $I = Ann(F)$, then $(I^{-1})_j$ is the $\mathbb{K}$-vector space spanned by the partial derivatives of order $\deg(F) - j$ of F.

One of the questions in the field is to determine F from the ideal I. It is not known how the shape of F makes the distinction between Artinian complete intersections and Artinian Gorenstein rings, as the first class is included in the second. On this idea, a question asked by Tony Geramita is if one can determine F from the minimal generators of an Artinian complete intersection ideal I, at the same time making this distinction. More generally, one would want to determine I^{-1} from I and conversely. Cho and Iarrobino [3] made some progress on this direction when I is the defining ideal of a zero-dimensional scheme in $\mathbb{P}^n$, saturated [3, Proposition 1.13] or locally Gorenstein [3, Theorem 3.3]. These results concern finding I^{-1} from the generators of I and conversely, finding the generators of I from I^{-1}.

Our notes follow the same direction: we determine F for the case when I is the Artinian reduction[1] of the ideal of a zero-dimensional reduced Gorenstein scheme (i.e., a Gorenstein set of points). Our Theorem 2.2 shows that F is determined uniquely from the homogeneous coordinates of the points that form this scheme: $F = \sum c_i L_i^r$, where L_i is the dual form of each point P_i in this set, r is the regularity, and c_i are non-zero constants, the sum being taken over all points in the set. This result resembles to [2, Theorem 3.8], the difference being that for a Gorenstein set of points, we specify who are the constants c_i in the decomposition $F = \sum c_i L_i^r$, and our approach is more homological than computational.[2]

Inverse systems occur naturally in the theory of systems of PDE's with constant coefficients, and similar results to the ones obtained via commutative algebraic methods appeared in the literature

1 By Artinian reduction we understand Artinian reduction by a general linear form.

2 Our main result in a way determines the inverse system from $V(I)$, rather than from I. A converse to this approach means to solve systems of multivariate polynomials that have a zero-dimensional saturated solution $V(I)$, by using inverse systems I^{-1} (see [15] for a detailed analysis).