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Examples of polycyclic groups with regular automorphisms of order 4

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ABSTRACT

We construct torsion-free polycyclic groups with fixed-point-free automorphisms of order 4 that are not nilpotent-by-finite, not metabelian-by-finite, not nilpotent-by-abelian and hence not centre-by-metabelian. (Note that such groups are always finite extensions of centre-by-metabelian groups.)

Let ϕ be an automorphism of the group G of order 4. If G is finite and if $C_G(\phi) = \langle 1 \rangle$, that is, if ϕ is fixed-point-free, then combining work of Kovács, see [3], and Gorenstein and Herstein, see [2], it follows that G is centre-by-metabelian. Moreover examples show that G need not be nilpotent nor metabelian. By Proposition 2.1 of Endimioni's paper [1] if $C_G(\phi)$ is finite and if G is polycyclic (more generally if G is a finite extension of a torsion-free soluble group of finite rank, see Corollary 1 of [4]), then G has a centre-by-metabelian characteristic subgroup of finite index. The finite counter examples referred to above do not rule out the possibilities that G here is nilpotent-by-finite or metabelian-by-finite. The following, however, does, even if we assume the stronger hypothesis that ϕ is fixed-point-free.

Example 1. There exists a torsion-free polycyclic group G with a fixed-point-free automorphism ϕ of order 4 such that G is not nilpotent-by-finite and is not metabelian-by-finite.

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If G is polycyclic-by-finite then G has a torsion-free characteristic subgroup H of finite index. If $C_G(\phi)$ is finite then $\phi|_H$ is fixed-point-free. We cannot conclude that H is centre-by-metabelian. In particular if $C_G(\phi) = \langle 1\rangle$, we can conclude that G is a finite extension of a centre-by-metabelian group, but not that G itself is centre-by-metabelian, unlike the finite case. We construct Example 2 below as an extension of a subgroup of Example 1.

Example 2. There exists a torsion-free polycyclic group S with a fixed-point-free automorphism ϕ of order 4 such that S is not nilpotent-by-abelian and in particular is not centre-by-metabelian.

Construction of Example 1. Let $K = \langle a, b\rangle$ be a free nilpotent group of class 2 and rank 2 on the basis $\{a, b\}$ and set $c = [b, a]$ (so $ba = abc$). Consider the automorphisms x and θ of K given by

$$x : a \mapsto b \quad \text{and} \quad b \mapsto a^{-1}b^3$$

and

$$\theta : a \mapsto a^{-1} \quad \text{and} \quad b \mapsto b^{-1}.$$

Then x acts rationally irreducibly and in particular fixed-point freely on K/K' (its eigenvalues on K/K' being $(3 \pm \sqrt{5})/2$) and centralizes $K' = \langle c\rangle$. Also $\theta^2 = 1$ and θ inverts K/K' and centralizes K'. Clearly x has infinite order, even on K/K'.

Let $k \mapsto k_i$ be an isomorphism of K onto the group K_i for $i = 1, 2$ and set $H = \langle K_1 \times K_2\colon\ c_1 = c_2^{-1}\rangle$, the central product of K_1 and K_2 amalgamating K_1' and K_2' via inversion. Let $y \in \operatorname{Aut} H$ act as x on K_1 and as x^{-1} on K_2; this map is well defined since x centralizes K'. Define the automorphism f of H by $(k_1 l_2)^f = (l\theta)_1 k_2$ for all k and l in K; again this is well defined. Also f^2 acts as θ on each of K_1 and K_2 and in particular inverts H/H' and centralizes H' and f has order 4.

Let $Y = \langle y^{\langle f\rangle}\rangle \leqslant \operatorname{Aut} H$ and $G = \langle Y, H\rangle$, a subgroup of the holomorph of H satisfying $[H', G] = \langle 1\rangle$. Then f normalizes G and defines an automorphism ϕ of G by conjugation; clearly ϕ has order 4. Now H' is isomorphic to the integers $\mathbf{Z}$ and $H/H' \cong \mathbf{Z}^{(4)}$. Also θ is central in $\operatorname{Aut}(K/K')$, so $y\phi = y^f$ acts as y^{-1} on H/H'. Thus $Y = \langle y\rangle Z$ for $Z = C_Y(H/H')$ and $Y/Z \cong \langle y\rangle \cong \mathbf{Z}$. Stability theory yields that Z embeds into $\operatorname{Hom}(\mathbf{Z}^{(4)}, \mathbf{Z})$, so Z is free abelian of rank at most 4. Therefore G is poly infinite cyclic (with Hirsch number between 6 and 10).

Now ϕ maps a_1, b_1, a_2, and b_2 to respectively a_2, b_2, a_1^{-1} and b_1^{-1} and these four elements determine a basis of the free abelian group H/H'. A simple check shows that ϕ acts fixed-point freely on H/H'. It also shows that $c_1\phi = c_2 = c_1^{-1}$ and H' is infinite cyclic; thus ϕ also acts fixed-point freely on H'. We saw above that ϕ acts fixed-point freely on Y/Z (it inverts it). Thus to prove that ϕ acts fixed-point freely on G it suffices to prove that ϕ acts fixed-point freely on Z.

Let $z \in Z$ and $h \in H$ and suppose $z\phi = z$. Then

$$[h.h\phi, z] = [h, z][h\phi, z] = [h, z][h, z]\phi = [h, z][h, z]^{-1} = 1.$$

Hence $C_H(z) \geqslant \langle h.h\phi\colon\ h \in H\rangle H'$. Let $k \in K$, so $k_i \in K_i$. Then $k_1(k_1\phi) = k_1k_2$ and $k_2(k_2\phi) = k_2(k\theta)_1 \in k_1^{-1}k_2H'$. Hence $k_2^2 \in C_H(z)$. Similarly $k_1^2 \in C_H(z)$ and therefore $H^2 \leqslant C_H(z)$. But then $[h, z]^2 = [h^2, z] = 1$ and yet H' is torsion-free. Consequently $[h, z] = 1$, so $[H, z] = \langle 1\rangle$ and yet $z \in \operatorname{Aut} H$. Hence $z = 1$ and therefore ϕ acts fixed-point freely on Z and G.

It remains to show that G is not nilpotent-by-finite and not metabelian-by-finite and for this it suffices to show that $\langle x\rangle K$ has neither of these properties. As pointed out above the eigenvalues of x on K/K' are $(3 \pm \sqrt{5})/2$. If $(3 + \sqrt{5})^m$ is rational for some positive integer m then $\sqrt{5}$ is rational, which it is not. Hence $\langle x^m\rangle$ acts rationally irreducibly on K/K' for every positive integer m. Suppose $\langle x\rangle K/K'$ is nilpotent-by-finite. Then x^m acts unipotently and hence rationally reducibly on K/K' for some positive integer m. Thus $\langle x\rangle K/K'$ and hence $\langle x\rangle K$ are not nilpotent-by-finite.