Journal of Algebra 400 (2014) 185–218

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Representing stable complexes on projective spaces

Jason Lo [a,1], Ziyu Zhang [b,*,2]

[a] Department of Mathematics, 202 Mathematical Sciences Building, University of Missouri, Columbia, MO 65211, USA

[b] Max-Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany

ARTICLE INFO

Article history:
Received 2 July 2012
Available online 22 December 2013
Communicated by V. Srinivas

MSC:
primary 14D20, 14F05
secondary 14J60

Keywords:
Reflexive sheaves
Polynomial stability
Quotient stacks
Bridgeland stability
Monads

ABSTRACT

We give an explicit proof of a Bogomolov-type inequality for c_3 of reflexive sheaves on $\mathbb{P}^3$. Then, using resolutions of rank-two reflexive sheaves on $\mathbb{P}^3$, we prove that the closed points of some strata of the moduli of rank-two complexes that are both PT-stable and dual-PT-stable can be given by the structure of quotient stacks. Using monads, we apply the same techniques to $\mathbb{P}^2$ and obtain similar results for some strata of the moduli of Bridgeland-stable complexes.

1. Introduction

Let X be any smooth projective threefold over k. In previous work [10, Section 4.2], we considered the moduli functor

$$\coprod_n \mathscr{M}^{\mathrm{PT}\cap\mathrm{PT}^*}_{(r,d,\beta,n)} \tag{1}$$

* Corresponding author.
E-mail addresses: jccl@alumni.stanford.edu (J. Lo), zz505@bath.ac.uk (Z. Zhang).
[1] Current address: Taimali, Taiwan.
[2] Current address: Department of Mathematics, University of Bath, Claverton Down, Bath, BA2 7AY, United Kingdom.

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2013.11.013

where the points $[E]$ of each moduli functor $\mathscr{M}^{\mathrm{PT}\cap\mathrm{PT}^*}_{(r,d,\beta,n)}$ represent complexes $E \in D^b(X)$ satisfying:

E is both PT-stable and PT-dual stable, and $ch(E) = (r, d, \beta, n)$.

We also observed in [10, Section 4.2] that, when r and d are both integers that are coprime, the points of (1) are in 1–1 correspondence with pairs of the form $([F], [Q^D])$, where

- $[F]$ is the isomorphism class of a μ-stable reflexive sheaf F on X with
$$\big(ch_0\big(F[1]\big), ch_1\big(F[1]\big), ch_2\big(F[1]\big)\big) = (r, d, \beta);$$
- $[Q^D]$ is the isomorphism class of the dual $Q^D := R\mathscr{H}om(Q, \mathscr{O}_X)[3]$ of Q, where Q is a quotient of the 0-dimensional sheaf $\mathscr{E}xt^1(F, \mathscr{O}_X)$.

Under this correspondence, we have $F = H^{-1}(E)$ and $Q^D = H^0(E)$ for any point $[E]$ of $\coprod_n \mathscr{M}^{\mathrm{PT}\cap\mathrm{PT}^*}_{(r,d,\beta,n)}$. We can therefore think of the moduli (1) as parametrising μ-stable reflexive sheaves F, with each isomorphism class occurring with multiplicity equal to the number of non-isomorphic quotients of $\mathscr{E}xt^1(F, \mathscr{O}_X)$.

This paper grew out of an attempt to find a more concrete and down-to-earth description of the objects parametrised by the moduli stack (1), with the hope that it might help us understand whether the stack (1) is a quotient stack. Objects in the derived category are often considered difficult to work with, because of the presence of quasi-isomorphisms (i.e. two very different-looking complexes can be isomorphic in the derived category). In Sections 3 and 4, we show that isomorphisms in the derived category for the objects in (1) can be understood as isomorphisms between diagrams in the category of coherent sheaves.

In Section 2, we give a Bogomolov-type inequality for μ-semistable reflexive sheaves F on $\mathbb{P}^3$. We point out, that it is already known that there is a bound for ch_3 in terms of ch_0, ch_1 and ch_2 for μ-semistable reflexive sheaves on a smooth variety over a field of characteristic zero. This is implicit in the proof of [12, Main Theorem], for example see also [7, Section 3] and [15]. However, we write down such an explicit bound for ch_3 in Theorem 2.1. The proof of Theorem 2.1 follows closely the ideas in [2], and is deferred to Section 5. As an immediate consequence of this theorem, we obtain that the moduli stack (1) is of finite type (Corollary 2.2).

In Section 3, we build on the work of Miró-Roig [13] and use particular 2-term locally free resolutions of reflexive sheaves to prove Theorem 3.6, which says that the closed points of certain strata of the moduli stack (1) are in bijection with the closed points of certain quotient stacks, when $X = \mathbb{P}^3$, $ch_0 = -2$ and $ch_1 = 1$.

In Section 4, we adapt the techniques in Section 3 from $\mathbb{P}^3$ to $\mathbb{P}^2$, using the results on monads due to Jardim [4] and Jardim–Martins [5]. This culminates in Theorem 4.9, that