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The Golod property for products and high symbolic powers of monomial ideals

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ABSTRACT

We show that for any two proper monomial ideals I and J in the polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$ the ring S/IJ is Golod. We also show that if I is squarefree then for large enough k the quotient $S/I^{(k)}$ of S by the kth symbolic power of I is Golod. As an application we prove that the multiplication on the cohomology algebra of some classes of moment-angle complexes is trivial.

1. Introduction

For a graded ideal I in the polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$ in n variables over the field $\mathbb{K}$ the ring S/I is called *Golod* if all Massey operations on the Koszul complex of S/I with respect of $\mathbf{x} = x_1, \ldots, x_n$ vanish. The naming gives credit to Golod [11] who

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showed that the vanishing of the Massey operations is equivalent to the equality case in the following coefficientwise inequality of power-series which was first derived by Serre:

$$\sum_{i\geqslant 0}\dim_{\mathbb{K}}\operatorname{Tor}_i^{S/I}(\mathbb{K},\mathbb{K})t^i \leqslant \frac{(1+t)^n}{1-t\sum_{i\geqslant 1}\dim_{\mathbb{K}}\operatorname{Tor}_i^{S}(S/I,\mathbb{K})t^i}$$

We refer the reader to [1] and [8] for further information on the Golod property and to [5] and [12] for the basic concepts from commutative algebra underlying this paper. We prove the following two results.

Theorem 1.1. *Let I, J be two monomial ideals in S different from S. Then S/IJ is Golod.*

For our results on symbolic powers we have to restrict ourselves to squarefree monomial ideals I. This is due to the fact that in this case I has a primary decomposition of the form $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$, where every $\mathfrak{p}_i$ is an ideal of S generated by a subset of the variables [12, Lem. 1.5.4]. Moreover, in this situation for a positive integer k the kth symbolic power $I^{(k)}$ of I coincides with $\mathfrak{p}_1^k \cap \cdots \cap \mathfrak{p}_r^k$ [12, Prop. 1.4.4].

Theorem 1.2. *Let I be a squarefree monomial ideal in S different from S. Then for $k \gg 0$ the kth symbolic power $I^{(k)}$ is Golod for $k \gg 0$.*

Besides the strong algebraic implications of Golodness the case of squarefree monomial ideals relates to interesting topology. Let Δ be a simplicial complex on ground set $[n]$ and let $\mathbb{K}[\Delta]$ be its Stanley–Reisner ring (see Section 4 for basic facts about Stanley–Reisner rings). By work of Buchstaber and Panov [6, Thm. 7.7], extending an additive isomorphism from [10], it is known that there is an algebra isomorphism of the Koszul homology $\mathrm{H}_*(\mathbf{x},\mathbb{K}[\Delta])$ and the singular cohomology ring $\mathrm{H}^*(M_\Delta;\mathbb{K})$ where $M_\Delta = \{(v_1,\ldots,v_n) \in (D^2)^n \mid \{i \mid v_i \notin S^1\} \in \Delta\}$. Here $D^2 = \{v \in \mathbb{R}^2 \mid \|v\| \leqslant 1\}$ is the unit disk in $\mathbb{R}^2$ and S^1 its bounding unit circle. Note that the isomorphism is not graded for the usual grading of $\mathrm{H}_*(\mathbf{x},\mathbb{K}[\Delta])$ and $\mathrm{H}^*(M_\Delta;\mathbb{K})$. The complex M_Δ is the *moment-angle complex* or *polyhedral product* of the pair (D^2,S^1) for Δ (we refer the reader to [6] and [7] for background information). Last we write $\Delta^\circ = \{A \subseteq [n] \mid [n]\setminus A \notin \Delta\}$ for the *Alexander dual* of the simplicial complex Δ. Now we are in position to formulate the following consequence of Theorem 1.1.

Corollary 1.3. *Let Δ be a simplicial complex such that $\Delta = (\Delta_1^\circ * \Delta_2^\circ)^\circ$ for two simplicial complexes Δ_1, Δ_2 on disjoint ground sets. Then the multiplication on $\mathrm{H}^*(M_\Delta;\mathbb{K})$ is trivial.*

The main tool for the proof of Theorem 1.1 and Theorem 1.2 is combinatorial. Let I be a monomial ideal and write $G(I)$ for the set of minimal monomial generators of I. In [19, Def. 3.8] the author introduces a combinatorial condition on $G(I)$ that in [3] was