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Amalgamation and interpolation in ordered algebras

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ABSTRACT

The first part of this paper provides a comprehensive and self-contained account of the interrelationships between algebraic properties of varieties and properties of their free algebras and equational consequence relations. In particular, proofs are given of known equivalences between the amalgamation property and the Robinson property, the congruence extension property and the extension property, and the flat amalgamation property and the deductive interpolation property, as well as various dependencies between these properties. These relationships are then exploited in the second part of the paper in order to provide new proofs of amalgamation and deductive interpolation for the varieties of lattice-ordered abelian groups and MV-algebras, and to determine important subvarieties of residuated lattices where these properties hold or fail. In particular, a full description is given of all subvarieties of commutative GMV-algebras possessing the amalgamation property.

1. Introduction

In Universal Algebra, a crucial and often extremely fruitful role is played by the fact that certain properties of a variety are "mirrored" by properties of their free algebras.

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In some cases, properties of free algebras may themselves be expressed as properties of associated equational consequence relations for the variety. The synthesis of these characterizations then provides an illuminating and potentially very useful "bridge" between the realms of algebra and logic.

A fundamental example of such a bridge is the relationship between the algebraic (or model-theoretic) property of amalgamation and the logical (or syntactic) property of interpolation. In this case, the amalgamation property for a variety is equivalent to the Robinson property for its consequence relations, which is equivalent in turn to a property of free algebras. These properties each imply the deductive interpolation property, which itself corresponds to an important property of free products: the flat amalgamation property. Conversely, the deductive interpolation property implies the amalgamation property in the presence of the congruence extension property or its syntactic equivalent, the extension property.

Relationships between these and other amalgamation, extension, and interpolation properties have received considerable attention in the literature. Publications of particular relevance to our discussion include Bacsich [4], Czelakowski and Pigozzi [14], Gabbay and Maksimova [22], Galatos and Ono [24], Kihara and Ono [43,44], Madarász [45], Maksimova [46–48], Montagna [53], Pierce [61], Pigozzi [62], Powell and Tsinakis [63–66], and Wroński [73,74]. We defer more precise historical and bibliographical details to the appropriate points in the text.

Our goal in the first part of this paper is to provide a comprehensive and self-contained presentation in a universal algebra setting of the most important interrelationships existing between amalgamation, interpolation, and extension properties. In contrast to the many other authors to have tackled these topics – in particular, the more general model-theoretic and abstract algebraic logic approaches of [4] and [14], respectively – we focus for clarity of exposition on varieties of algebras and make use only of quite basic concepts from universal algebra in developing our account. The result is a more direct and accessible (but of course more restricted in scope) presentation of the topics. A further novelty of our approach is that we emphasize the fundamental role played by the equational consequence relation of a variety (relative to equations defined over a fixed countably infinite set of variables), thereby obtaining equivalent formulations of algebraic properties restricted to countable algebras.

The broad goal of the second part of the paper is to make use of the afore-mentioned relationships in order to investigate amalgamation and interpolation properties for specific varieties of ordered algebras. We first provide new "syntactic" proofs of the amalgamation property for abelian lattice-ordered groups and MV-algebras. We then turn our attention to varieties of residuated lattices, a framework that provides algebraic semantics for substructural logics as well as covering other important classes of algebras such as lattice-ordered groups. We study in some depth the amalgamation property for subvarieties of GBL-algebras; in particular, we provide a full description of all subvarieties of commutative GMV-algebras that have the amalgamation property.