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On cohomology of saturated fusion systems and support varieties

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ABSTRACT

In this short note we study the cohomology algebra of saturated fusion systems using finite groups which realize saturated fusion systems and Hochschild cohomology of group algebras. A similar result to a theorem of Alperin [1] is proved for varieties of cohomology algebras of fusion systems associated to block algebras of finite groups.

1. Introduction

A saturated fusion system $\mathscr{F}$ on a finite p-group P is a category whose objects are the subgroups of P and whose morphisms satisfy certain axioms mimicking the behavior of a finite group G having P as a Sylow subgroup. The axioms of saturated fusion systems were invented by Puig in early 1990's. The cohomology algebra of a p-local finite

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group with coefficients in $\mathbb{F}_p$ is introduced in [4, §5]. Let k be an algebraically closed field of characteristic p. We denote by $\mathrm{HH}^*(kG)$ the Hochschild cohomology algebra of the group algebra kG and by $\mathrm{H}^*(G,k)$ the cohomology algebra of the group G with trivial coefficients. As in [8] we will use the language of homotopy classes of chain maps (see [8, 2.8, 4.2]). We denote by $\mathrm{H}^*(\mathscr{F})$ the algebra of stable elements of $\mathscr{F}$, i.e. the cohomology algebra of the saturated fusion system $\mathscr{F}$, which is the subalgebra of $\mathrm{H}^*(P,k)$ consisting of elements $[\zeta] \in \mathrm{H}^*(P,k)$ such that

$$\mathrm{res}^P_Q\bigl([\zeta]\bigr) = \mathrm{res}_\varphi\bigl([\zeta]\bigr),$$

for any $\varphi \in \mathrm{Hom}_{\mathscr{F}}(Q,P)$ and any subgroup Q of P. This is the main object of study in this paper. Moreover Broto, Levi and Oliver showed that any saturated fusion system $\mathscr{F}$ has a non-unique P–P-biset X with certain properties formulated by Linckelmann and Webb (see [4, Proposition 5.5]). Such a P–P-biset X is called a characteristic biset. Using this biset, S. Park noticed in [11] a result which says that a saturated fusion system can be realized by a finite group. This finite group is $G = \mathrm{Aut}(X_P)$, that is the group of bijections of the characteristic biset X, preserving the right P-action. So, by [11, Theorem 3], we can identify $\mathscr{F}$ with $\mathscr{F}_P(G)$ which is the fusion system on P such that for every $Q, R \leqslant P$ we have

$$\mathrm{Hom}_{\mathscr{F}_P(G)}(Q,R) = \bigl\{\varphi\colon\ Q \to R \bigm| \exists x \in G \text{ s.t. } \varphi(u) = xux^{-1},\ \forall u \in Q\bigr\}.$$

In Section 2 for a finite group G and P a p-subgroup such that $\mathscr{F}_P(G)$ is a saturated fusion system we associate and analyze a restriction map from the cohomology algebra of the group G with coefficients in the field k to the cohomology algebra of the fusion system, $\mathrm{H}^*(\mathscr{F}_P(G))$. For shortness we denote $\mathscr{F} = \mathscr{F}_P(G)$ and this map by $\rho_{\mathscr{F},G}$. From the above we can identify any saturated fusion system $\mathscr{F}$ with a saturated fusion system of the form $\mathscr{F}_P(G)$, where $G = \mathrm{Aut}(X_P)$ for X a characteristic biset; but we prefer to work under a more general setup. Although some results are straightforward translations of results in the literature, to prove them we need the machinery of transfer maps between Hochschild cohomology algebras of group algebras developed in [8]. Since $\mathrm{H}^*(\mathscr{F})$ is a graded commutative finitely generated k-algebra we can associate the spectrum of maximal ideals, i.e. the variety denoted $V_{\mathscr{F}}$. For U a finitely generated kP-module we define also a support variety, denoted $V_{\mathscr{F}}(U)$, in a similar way as the usual support variety in group cohomology. The main result of this section is Theorem 2.1, where we prove that $\rho_{\mathscr{F},G}$ induces a finite map on varieties and that we can recover $V_{\mathscr{F}}(U)$ from the support variety $V_G(U)$ associated to any finitely generated kG-module U, which was first introduced by Carlson in [6].

In Section 3 we consider only fusion systems associated to blocks. If b is a block idempotent of the group algebra kG (i.e. a primitive idempotent in the center of the group algebra) with defect group P then it is well known that the set of b-Brauer pairs form a G-poset which defines a saturated fusion system. We denote this category by $\mathscr{F}_{G,b}$.