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Trivial unit conjecture and homotopy theory

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ABSTRACT

A homotopy theoretic description is given for trivial unit conjecture in the group ring $\mathbb{Z}G$.

1. Introduction

Let G be a torsion-free group and $\mathbb{Z}G$ the integral group ring. The trivial unit conjecture for G says that any invertible element (unit) of $\mathbb{Z}G$ is of the form $\pm g$ for some $g \in G$ (cf. [6], Chapter 13). For solving such a conjecture, to the author's knowledge, almost all the approaches used are algebraic (cf. [1] and references therein). In this note, we give a homotopy theoretic description of such a conjecture.

Let X be a CW complex with fundamental group $\pi_1(X) = G$. For any integer $d \geqslant 2$ and map $f : S^d \to X \vee S^d$, we construct a CW complex $Y_f = (X \vee S^d) \cup_f e^{d+1}$. In this note, the following homotopy theoretic characterization is obtained:

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Theorem 1. *Let G be a torsion-free group. The trivial unit conjecture for G is true if and only if for an Eilenberg–Mac Lane space $X = BG$, the element $[f] \in \pi_d(\widetilde{X \vee S^d}, S^d)$ (the relative homotopy group of the universal covering space) vanishes for some lifting of S^d whenever the inclusion $i_f : X \to Y_f$ is a homotopy equivalence.*

All modules considered in this note are left modules. Let $\tilde{Y}_f$ be the universal covering space of Y_f and $C_i(\tilde{Y}_f)$ the i-th term of the cellular chain complex of $\tilde{Y}_f$. By definition, $C_i(\tilde{Y}_f)$ is a free $\mathbb{Z}G$-module spanned by the set of all i-cells. For the inclusion $i_f : X \to Y_f$, we have a cellular map $\tilde{i}_f : \tilde{X} \to \tilde{Y}_f$ which lifts i_f. As the map i_f induces the identity homomorphism on fundamental groups of X and Y_f, we may assume that $\tilde{X}$ is a subspace of $\tilde{Y}_f$. The relative chain complex $C_*(\tilde{Y}_f, \tilde{X})$ of $(\tilde{Y}_f, \tilde{X})$ is of the following form

$$0 \to C_{d+1}(\tilde{Y}_f, \tilde{X}) = \mathbb{Z}G \stackrel{\partial}{\to} C_d(\tilde{Y}_f, \tilde{X}) = \mathbb{Z}G \to 0.$$

This is a chain complex whose terms are all vanishing except for the d-th term a free $\mathbb{Z}G$-module spanned by S^d and the $(d+1)$-th term a free $\mathbb{Z}G$-module spanned by e^{d+1}. Let $\gamma_f = \partial(1) \in \mathbb{Z}G$, the unique element determined by the boundary map ∂. We give a homotopy theoretic description of units in $\mathbb{Z}G$ as follows.

Lemma 2. *Let $\gamma_f \in \mathbb{Z}G$ be the element defined above. Then γ_f is an invertible element if and only if the inclusion $i_f : X \hookrightarrow Y_f$ is a homotopy equivalence.*

Proof. All the notations used in this proof are the same as defined before. Suppose that $\gamma_f = \partial(1)$ is an invertible element in $\mathbb{Z}G$. Then ∂ is both injective and surjective, which shows the relative chain complex $C_*(\tilde{Y}_f, \tilde{X})$ is acyclic. This implies that $\tilde{i}_f$ induces an isomorphism between the homology groups $H_i(\tilde{X})$ and $H_i(\tilde{Y}_f)$ for each $i \geqslant 0$. Since $\tilde{X}$ and $\tilde{Y}_f$ are both simply connected, $\tilde{i}_f : \tilde{X} \to \tilde{Y}_f$ is a homotopy equivalence. Since i_f induces the identity homomorphism on fundamental groups, this shows that $i_f : X \to Y_f$ is a homotopy equivalence by the Whitehead theorem.

Conversely, suppose that $i_f : X \to Y_f$ is a homotopy equivalence. Then $\tilde{i}_f : \tilde{X} \to \tilde{Y}_f$ is a homotopy equivalence, which implies that the relative chain complex $C_*(\tilde{Y}_f, \tilde{X})$ is acyclic. This implies that $\gamma_f = \partial(1)$ has a left inverse. It is a well-known fact that in the integral group ring of a torsion-free group, one-sided invertible element is also two-sided invertible (cf. Corollary 1.9 from [6, p. 38]). This finishes the proof. □

Proof of Theorem 1. Let $X = BG$, the classifying space of G. Suppose that the trivial unit conjecture for G is true. For an integer $d \geqslant 2$ and a map $f : S^d \to X \vee S^d$, suppose that the CW complex $Y_f = (X \vee S^d) \cup_f e^{d+1}$ has its inclusion $i_f : X \to Y_f$ a homotopy equivalence. By Lemma 2, the element γ_f is a unit. Therefore, $\gamma_f = \pm g$ for some element $g \in G$. As the d-th and $(d+1)$-th terms of the relative chain complex are free $\mathbb{Z}G$-modules, we can view them as submodules of $C_i(\tilde{Y})$ $(i = d, d+1$ resp.). Since $\tilde{X}$ is a free G-CW complex and S^d is simply connected, the universal covering space $\widetilde{X \vee S^d}$ could be taken as the push out the following diagram