Journal of Algebra 402 (2014) 319–350

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Weak crossed-product orders over valuation rings

John S. Kauta

School of Computing, Information & Mathematical Sciences, The University of the South Pacific, Suva, Fiji Islands

A R T I C L E I N F O

Article history:
Received 25 November 2012
Available online 11 January 2014
Communicated by J.T. Stafford

MSC:
13F30
16E60
16H05
16H10
16S35

Keywords:
Crossed-product orders
Weak crossed-product orders
Semihereditary orders
Hereditary orders
Extremal orders
Maximal orders
Azumaya algebras
Dubrovin valuation rings

A B S T R A C T

Let F be a field, let V be a valuation ring of F of arbitrary Krull dimension (rank), let K be a finite Galois extension of F with group G, and let S be the integral closure of V in K. Let $f : G \times G \mapsto K \setminus \{0\}$ be a normalized two-cocycle such that $f(G \times G) \subseteq S \setminus \{0\}$, but we do not require that f should take values in the group of multiplicative units of S. One can construct a crossed-product V-order $A_f = \sum_{\sigma \in G} Sx_\sigma$ with multiplication given by $x_\sigma s x_\tau = \sigma(s)f(\sigma,\tau)x_{\sigma\tau}$ for $s \in S$, $\sigma,\tau \in G$. We characterize semihereditary and Dubrovin crossed-product orders, under mild valuation-theoretic assumptions placed on the nature of the extension K/F.

1. Introduction

This work is about orders in a crossed-product F-algebra $(K/F, G, f)$ which are integral over their center V, a valuation ring of the field F, and generalizes the results obtained in [4,12,13,25]. Whereas in [4,12,13] the valuation ring V is unramified in K, and in [4,12,25] V is a discrete valuation ring (DVR), we seldomly make such assumptions in this paper. Further, our theory complements and enhances the classical theory of crossed-product orders over valuation rings, such as can be found in [6,23], in two

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2013.12.014

significant ways: firstly, we do not require that the valuation ring V should be a DVR, as mentioned above; secondly, we do not require that the values of the two-cocycles associated with our crossed-product orders should be multiplicative units in the integral closure of V in K. We shall give a precise description of the crossed-product orders we will be dealing with shortly.

If R is a ring, then $J(R)$ will denote its Jacobson radical, $Z(R)$ its center, $U(R)$ its group of multiplicative units, and $R^{\#}$ the subset of all the non-zero elements. The residue ring $R/J(R)$ will be denoted by $\overline{R}$. Given the ring R, it is called *primary* if $J(R)$ is a maximal ideal of R. It is called *hereditary* if one-sided ideals are projective R-modules. It is called *semihereditary* (respectively *Bézout*) if finitely generated one-sided ideals are projective R-modules (respectively are principal). Let V be a valuation ring of a field F. If Q is a finite-dimensional central simple F-algebra, then a subring R of Q is called an order in Q if $RF = Q$. If in addition $V \subseteq R$ and R is integral over V, then R is called a V-order. If a V-order R is maximal among the V-orders of Q with respect to inclusion, then R is called a *maximal V-order* (or just a *maximal order* if the context is clear). A V-order R of Q is called an *extremal* V-order (or simply *extremal* when the context is clear) if for every V-order B in Q with $B \supseteq R$ and $J(B) \supseteq J(R)$, we have $B = R$. If R is an order in Q then, since it is a PI-ring, it is called a *Dubrovin valuation ring* of Q (or a *valuation ring* of Q in short) if it is semihereditary and primary (see [2]).

In this paper, V will denote a commutative valuation domain of *arbitrary* Krull dimension (rank). Let F be its field of quotients, let K/F be a finite Galois extension with group G, and let S be the integral closure of V in K. If $f \in Z^2(G, U(K))$ is a normalized two-cocycle such that $f(G \times G) \subseteq S^{\#}$, then one can construct a "crossed-product" V-algebra

$$A_f = \sum_{\sigma \in G} S x_\sigma,$$

with the usual rules of multiplication: $x_\sigma s x_\tau = \sigma(s) f(\sigma, \tau) x_{\sigma\tau}$ for $s \in S$, $\sigma, \tau \in G$. Then A_f is associative, with identity $1 = x_1$, and center $V = V x_1$. Further, A_f is a V-order in the crossed-product F-algebra $\Sigma_f = \sum_{\sigma \in G} K x_\sigma = (K/F, G, f)$.

Two such cocycles f and g are said to be cohomologous over S (respectively cohomologous over K), denoted by $f \sim_S g$ (respectively $f \sim_K g$), if there are elements $\{c_\sigma \mid \sigma \in G\} \subseteq U(S)$ (respectively $\{c_\sigma \mid \sigma \in G\} \subseteq K^{\#}$) such that $g(\sigma, \tau) = c_\sigma \sigma(c_\tau) c_{\sigma\tau}^{-1} f(\sigma, \tau)$ for all $\sigma, \tau \in G$. Following [4], let $H = \{\sigma \in G \mid f(\sigma, \sigma^{-1}) \in U(S)\} = \{\sigma \in G \mid x_\sigma \in U(A_f)\}$. Then H is a subgroup of G. On G/H, the left coset space of G by H, one can define a partial ordering by the rule $\sigma H \leqslant \tau H$ if $f(\sigma, \sigma^{-1}\tau) \in U(S)$. Then "$\leqslant$" is well-defined, and depends only on the cohomology class of f over S. Further, H is the unique least element. We call this partial ordering on G/H *the graph of f* and in this paper we will denote it by $\mathrm{Gr}(f)$.

Such a setup was first formulated by Haile in [4], with the assumption that V is a DVR unramified in K, wherein, among other things, conditions equivalent to such orders being