Journal of Algebra 402 (2014) 379–391

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Palindromic width of free nilpotent groups ☆

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ARTICLE INFO

Article history:
Received 16 February 2013
Available online 14 January 2014
Communicated by E.I. Khukhro

MSC:
primary 20F18
secondary 20D15, 20E05

Keywords:
Palindromic width
Free nilpotent groups

ABSTRACT

In this paper we consider the palindromic width of free nilpotent groups. In particular, we prove that the palindromic width of a finitely generated free nilpotent group is finite. We also prove that the palindromic width of a free abelian-by-nilpotent group is finite.

1. Introduction

Let G be a group with a set of generators A. A reduced word in the alphabet $A^{\pm 1}$ is a *palindrome* if it reads the same forwards and backwards. The palindromic length $l_{\mathcal{P}}(g)$ of an element g in G is the minimum number k such that g can be expressed as a product of k palindromes. The *palindromic width* of G with respect to A is defined to be $\mathrm{pw}(G) = \sup_{g \in G} l_{\mathcal{P}}(g)$. In analogy with commutator width of groups (for example see [2–5]), it is a problem of potential interest to study palindromic width of groups. Palindromes

☆ The authors gratefully acknowledge the support of the Indo-Russian DST-RFBR project grant DST/INT/RFBR/P-137.
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0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2013.12.002

of free groups have already proved useful in studying various aspects of combinatorial group theory, for example see [8,9,12]. In [6], it was proved that the palindromic width of a non-abelian free group is infinite. This result was generalized in [7] where the authors proved that almost all free products have infinite palindromic width; the only exception is given by the free product of two cyclic groups of order two, when the palindromic width is two. Piggot [13] studied the relationship between primitive words and palindromes in free groups of rank two. It follows from [6,13] that up to conjugacy, a primitive word can always be written as either a palindrome or a product of two palindromes and that certain pairs of palindromes will generate the group. Recently Gilman and Keen [10,11] have used tools from hyperbolic geometry to reprove this result and further have obtained discreteness criteria for two generator subgroups in $\mathrm{PSL}(2,\mathbb{C})$ using the geometry of palindromes. The work of Gilman and Keen indicates a deep connection between palindromic width of groups and geometry.

Let $\mathrm{N}_{n,r}$ be the free r-step nilpotent group of rank n. In this paper we consider the palindromic width of free nilpotent groups. We prove that the palindromic width of a finitely generated free nilpotent group is finite. In fact, we prove that the palindromic width of an arbitrary rank n free nilpotent group is bounded by $3n$. For the 2-step free nilpotent groups, we improve this bound. For the groups, $\mathrm{N}_{n,1}$ and $\mathrm{N}_{2,2}$ we get the exact values of the palindromic width. Our main theorem is the following.

Theorem 1.1. *Let $\mathrm{N}_{n,r}$ be the free r-step nilpotent group of rank $n \geqslant 2$. Then the following holds*:

(1) *The palindromic width* $\mathrm{pw}(\mathrm{N}_{n,1})$ *of a free abelian group of rank n is equal to n.*
(2) *For $r \geqslant 2$, $n \leqslant \mathrm{pw}(\mathrm{N}_{n,r}) \leqslant 3n$.*
(3) $\mathrm{pw}(\mathrm{N}_{n,2}) \leqslant 3(n-1)$.

We prove the theorem in Section 3. Along the way, we also prove that $\mathrm{pw}(\mathrm{N}_{2,2}) = 3$. In Section 2, after recalling some basic notions and related basic results, we prove Lemma 2.5 which is a key ingredient in the proof of Theorem 1.1. As a consequence of Lemma 2.5 we also prove that the palindromic width of a free abelian-by-nilpotent group of rank n is bounded by $5n$, see Proposition 3.7. For the group $\mathrm{N}_{3,2}$ it is possible to improve the bound given in (2) of the above theorem. In fact, $4 \leqslant \mathrm{pw}(\mathrm{N}_{3,2}) \leqslant 6$. A detailed proof of this fact will appear elsewhere. It would be interesting to obtain solutions to the following problems.

Problem 1.

(1) For $n \geqslant 3$, $r \geqslant 2$, find $\mathrm{pw}(\mathrm{N}_{n,r})$.
(2) Construct an algorithm that determines $l_{\mathcal{P}}(g)$ for arbitrary $g \in \mathrm{N}_{n,r}$.