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Generalised norms in finite soluble groups

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We give a framework for a number of generalisations of Baer's norm that have appeared recently. For a class $\mathfrak{C}$ of finite nilpotent groups we define the $\mathfrak{C}$-norm $\kappa_{\mathfrak{C}}(G)$ of a finite group G to be the intersection of the normalisers of the subgroups of G that are not in $\mathfrak{C}$. We show that those groups for which the $\mathfrak{C}$-norm is not hypercentral have a very restricted structure. The non-nilpotent groups G for which $G = \kappa_{\mathfrak{C}}(G)$ have been classified for some classes. We give a classification for nilpotent classes closed under subgroups, quotients and direct products of groups of coprime order and show the known classifications can be deduced from our classification.

1. Introduction

All groups we consider are finite. Baer [1] defined the norm $\kappa(G)$ of a group G to be the intersection of the normalisers of all subgroups. Schenkman [15] later proved that the norm is always contained in the second centre. He also stated that its centraliser contains the commutator subgroup of the group, but the proof does not work. A correct proof is in the Zentralblatt review of this paper, given by W. Kappe [7].[1] Some generalisations

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[1] We are grateful to the referee for giving such information to us.

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of Baer's norm have been considered in the papers [14,16,17]. Our aim here is to show that in many cases the generalised norm will be close to being hypercentral and groups with non-hypercentral generalised norm have a restricted structure.

We assume throughout that $\mathfrak{C}$ is a class of groups, closed under subgroups, quotient groups and direct products of groups of coprime order. We define the $\mathfrak{C}$-norm $\kappa_{\mathfrak{C}}(G)$ of a group G to be the intersection of the normalisers of the subgroups of G that are not in $\mathfrak{C}$: $\kappa_{\mathfrak{C}}(G) = \bigcap_{H \notin \mathfrak{C}} N_G(H)$ with the convention that $\kappa_{\mathfrak{C}}(G) = G$ if $G \in \mathfrak{C}$. (Baer's norm can be thought of as $\kappa_{\mathfrak{E}}(G)$ where $\mathfrak{E}$ is the class of groups of order 1.) Note that $\kappa_{\mathfrak{C}}(G)$ is a characteristic subgroup of G. We will be interested in the properties of $\kappa_{\mathfrak{C}}(G)$ and also in the structure of groups G for which $\kappa_{\mathfrak{C}}(G) = G$. For a general $\mathfrak{C}$, a $\mathfrak{C}$-group G always satisfies $\kappa_{\mathfrak{C}}(G) = G$ and so we will only consider groups that are not $\mathfrak{C}$-groups. For Baer's norm these are just the Dedekind groups, a well-known class. A group G for which $\kappa_{\mathfrak{C}}(G) = G$ will be called a $\mathfrak{C}$-Dedekind group. The structure of non-nilpotent $\mathfrak{C}$-Dedekind groups has been considered for a number of classes: $\mathfrak{Z}$ (the class of cyclic groups) (classified by Shen, Shi and Zhang [16]), $\mathfrak{A}$ (the class of abelian groups) (first studied by Romalis and Sesekin [14] and classified by Nagrebeckii [9]), $\mathfrak{N}$ (the class of nilpotent groups) (classified by Nagrebeckii [10] and Bruno and Phillips [4]). The nilpotent $\mathfrak{C}$-Dedekind groups require different techniques to those used in this paper. They have been classified for $\mathfrak{Z}$ (the class of cyclic groups) (classified by Passman [11] and Bozikov and Janko [2]; see also [13]), $\mathfrak{A}$ (the class of abelian groups) (classified by Romalis and Sesekin [14]). The $\mathfrak{A}$-Dedekind groups were called meta-Hamiltonian groups by Romalis and Sesekin.

We consider classes $\mathfrak{C} \subseteq \mathfrak{N}$. If $\kappa_{\mathfrak{C}}(G)$ contains a non-central chief factor of G, we show that $\kappa_{\mathfrak{C}}(G)$ contains exactly one non-central chief factor (in a chief series of G through $\kappa_{\mathfrak{C}}(G)$) and if p is the prime dividing the order of this chief factor then a Hall p'-subgroup of G is a $\mathfrak{C}$-group. Theorem 1 gives more detailed structure, showing in particular that G has nilpotent length at most 3. If $\kappa_{\mathfrak{C}}(G)$ is non-nilpotent, Theorem 2 shows the structure of G is further restricted. We give a detailed description of the structure of G and note that as a consequence for some set of primes ρ (with $p \notin \rho$) G is the central product of a $\mathfrak{C}$-Dedekind $\rho \cup \{p\}$-group and a ρ'-group. We also give a classification theorem for $\mathfrak{C}$-Dedekind groups. We show that the known classifications can be deduced from ours (Theorems 3, 4 and 5).

It is easy to construct examples of groups G for which $\kappa_{\mathfrak{C}}(G)$ contains a non-central chief factor (S_3 is a $\mathfrak{C}$-Dedekind group for any class $\mathfrak{C}$ that contains cyclic a group of order 6). In the final section we give examples to show the types of non-central chief factors that can arise.

2. Preliminaries

We begin with an easy lemma.

Lemma 1. *For any group G and class $\mathfrak{C}$, $\kappa(G) \leqslant \kappa_{\mathfrak{C}}(G)$.*

The proof is obvious.