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How to obtain division algebras from a generalized Cayley–Dickson doubling process

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ABSTRACT

New families of eight-dimensional real division algebras with large derivation algebra are presented: We generalize the classical Cayley–Dickson doubling process, starting with a quaternion algebra over a field F and allowing the element used in the doubling to be an invertible element in the algebra. The resulting unital algebras are not third power-associative, hence not quadratic. Starting with a quaternion division algebra D, we obtain division algebras A for all elements chosen in D outside of F. This is independent of where the element is placed inside the product. Thus three pairwise non-isomorphic families of eight-dimensional division algebras are obtained. Their Albert isotopes yield more division algebras with large derivation algebra.

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Introduction

It is well-known that every real division algebra must have dimension 1, 2, 4 or 8 [21,17] and, indeed, the same result applies to division algebras over a real closed field [10].

Real division algebras were roughly classified by Benkart and Osborn [5,6] according to the isomorphism type of their derivation algebra. With respect to the algebras constructed in [6], it is noted in the introduction of [6]: “as one might expect, most of the

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classes of division algebras are natural generalizations of the quaternions and octonions," with the only exception of one family of flexible algebras which includes Okubo's pseudo-octonion algebras. The authors constructed division algebras over fields of characteristic 0 by slightly manipulating the usual octonion multiplication. For eight-dimensional real division algebras, 5 different types of possible derivation algebras can occur. Using the representation theory of Lie algebras, for each type of derivation algebra families of real division algebras of dimension 8 were investigated which display this non-zero Lie algebra as their derivation algebra [5,6]. This work was continued in [24] and [14,15].

Flexible real division algebras were classified in subsequent papers by Benkart, Britten, Osborn [4], Cuenca Mira, De Los Santos Villodres, Kaidi and Rochdi [7] and Darpö [8,9]. Ternary derivations were used in [16] to describe large classes of division algebras. Power-associative real division algebras of dimension less or equal to 4 were classified in [13], to list just of a few of the other known results.

The problem to find all real division algebras which are not flexible remains open.

Let F be a field. A *nonassociative quaternion algebra* over F is a four-dimensional unital F-algebra A whose nucleus is a quadratic étale algebra over F. Nonassociative quaternion division algebras canonically appeared as the most interesting case in the classification of the algebras of dimension 4 over F which contain a separable quadratic field extension of F in their nucleus (Waterhouse [28], see also Althoen, Hansen and Kugler [2] for $F = \mathbb{R}$). They were first discovered by Dickson [11] in 1935 and Albert [1] in 1942 as early examples of real division algebras. Waterhouse's classification shows that every nonassociative quaternion algebra is a Cayley–Dickson doubling of its nucleus S, where the element chosen for this doubling process is an invertible element in S not contained in the base field, see also [3].

In this paper, we extend this idea and construct algebras over F using the Cayley–Dickson doubling of a unital algebra with involution, where the element chosen for the doubling process is an invertible element of the algebra which is not contained in the base field. There are now three different possibilities how to place the element c used in the doubling process. More precisely, given an algebra D over F with an involution $\sigma : D \to D$ and an invertible element $c \in D$ such that $\sigma(c) \neq c$, the F-vector space $D \oplus D$ is made into an algebra over F via the multiplications

(1) $(u, v)(u', v') = (uu' + c(\sigma(v')v), v'u + v\sigma(u'))$,
(2) $(u, v)(u', v') = (uu' + \sigma(v')(cv), v'u + v\sigma(u'))$,
or
(3) $(u, v)(u', v') = (uu' + (\sigma(v')v)c, v'u + v\sigma(u'))$

and the corresponding algebras are denoted by $\mathrm{Cay}(D, c)$, $\mathrm{Cay}_m(D, c)$ and $\mathrm{Cay}_r(D, c)$, respectively.

The description of these algebras is straightforward and not based on structure coefficients with respect to some basis. They are unital, not power-associative and not quadratic. They do not even satisfy the weaker property that $u^2u = uu^2$ holds for

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