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Quantum Schur–Weyl duality and projected canonical bases

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A B S T R A C T

Let $\mathscr{H}_r$ be the generic type A Hecke algebra defined over $\mathbb{Z}[u,u^{-1}]$. The Kazhdan–Lusztig bases $\{C_w\}_{w\in\mathcal{S}_r}$ and $\{C'_w\}_{w\in\mathcal{S}_r}$ of $\mathscr{H}_r$ give rise to two different bases of the Specht module M_λ, $\lambda\vdash r$, of $\mathscr{H}_r$. These bases are not equivalent and we show that the transition matrix $S(\lambda)$ between the two is the identity at $u=0$ and $u=\infty$. To prove this, we first prove a similar property for the transition matrices $\tilde{T}$, $\tilde{T}'$ between the Kazhdan–Lusztig bases and their projected counterparts $\{\tilde{C}_w\}_{w\in\mathcal{S}_r}$, $\{\tilde{C}'_w\}_{w\in\mathcal{S}_r}$, where $\tilde{C}_w := C_w p_\lambda$, $\tilde{C}'_w := C'_w p_\lambda$ and p_λ is the minimal central idempotent corresponding to the two-sided cell containing w. We prove this property of $\tilde{T},\tilde{T}'$ using quantum Schur–Weyl duality and results about the upper and lower canonical basis of $V^{\otimes r}$ (V the natural representation of $U_q(\mathfrak{gl}_n)$) from [14,11,7]. We also conjecture that the entries of $S(\lambda)$ have a certain positivity property.

1. Introduction

Let $\{C_w\colon\ w\in\mathcal{S}_r\}$ and $\{C'_w\colon\ w\in\mathcal{S}_r\}$ be the Kazhdan–Lusztig bases of the type A Hecke algebra $\mathscr{H}_r$, which we refer to as the upper and lower canonical basis of $\mathscr{H}_r$, respectively. After working with these bases for a while, we have convinced ourselves that it is not particularly useful to look at both at once—one can work with one or the other and it is easy to go back and forth between the two (precisely, there is an automorphism

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θ of $\mathscr{H}_r$ such that $\theta(C'_w) = (-1)^{\ell(w)} C_w$). However, our recent work on the nonstandard Hecke algebra [4,6] has forced us to look at both these bases simultaneously. Before explaining how this comes about, let us describe our results and conjectures.

Let $K = \mathbb{Q}(u)$, where u is the Hecke algebra parameter, and let M_λ be the $K\mathscr{H}_r$-irreducible of shape $\lambda \vdash r$. The upper and lower canonical basis of $\mathscr{H}_r$ give rise to bases $\{C_Q\colon Q \in \mathrm{SYT}(\lambda)\}$ and $\{C'_Q\colon Q \in \mathrm{SYT}(\lambda)\}$ of M_λ, which we refer to as the upper and lower canonical basis of M_λ. These bases are not equivalent, and it appears to be a difficult and interesting question to understand the transition matrix $S(\lambda)$ between them (which is well-defined up to a global scale by the irreducibility of M_λ). It turns out that $S(\lambda)$ is the identity at $u = 0$ and $u = \infty$ (Theorem 7.8) and, though it is not completely clear what it should mean for an element of K to be nonnegative, its entries appear to have some kind of nonnegativity (see Conjecture 7.9).

To compare the upper and lower canonical basis of M_λ, we compare them both to certain seminormal bases of M_λ in the sense of [28]. These bases are compatible with restriction along the chain of subalgebras $\mathscr{H}_1 \subseteq \cdots \subseteq \mathscr{H}_{r-1} \subseteq \mathscr{H}_r$ (see Definition 7.3). Specifically, we define an upper (resp. lower) seminormal basis which differs from the upper (resp. lower) canonical basis by a unitriangular transition matrix $T(\lambda)$ (resp. $T'(\lambda)$). It appears that these transition matrices also possess some kind of nonnegativity property. Since the restrictions $\mathscr{H}_{i-1} \subseteq \mathscr{H}_i$ are multiplicity-free, these seminormal bases differ from each other by a diagonal transformation $D(\lambda)$. Hence we have $S(\lambda) = T(\lambda)D(\lambda)T'(\lambda)^{-1}$.

We briefly mention some related investigations in the literature. Other seminormal bases of M_λ have been defined—for instance, Hoefsmit, and later, independently, Ocneanu, and Wenzl construct a Hecke algebra analog of Young's orthogonal basis (see [30]). This basis differs from our upper and lower seminormal bases by a diagonal transformation, but is not equal to either. The recent paper [8] uses an interpretation of the lower seminormal basis in terms of non-symmetric Macdonald polynomials to study $T'(\lambda)$ for λ a two-row shape and gives an explicit formula for a column of this matrix (see Remark 8.4). Along similar lines, the transition matrix between the upper canonical basis at $u = 1$ and Young's natural basis of M_λ is studied by Garsia and McLarnan in [13]; they show that this matrix is unitriangular and has integer entries.

Our investigation further involves projecting the basis element C_w (resp. C'_w) onto the isotypic component corresponding to the two-sided cell containing w. This results in what we call the projected upper (resp. lower) canonical basis; let $\tilde{T}$ (resp. $\tilde{T}'$) denote the transition matrix between the projected and upper (resp. lower) canonical basis. The properties we end up proving about $S(\lambda), T(\lambda), T'(\lambda)$ all follow from properties of $\tilde{T}$ and $\tilde{T}'$. And we are able to get some handle on $\tilde{T}$ and $\tilde{T}'$ using quantum Schur–Weyl duality. Specifically, we use the compatibility between an upper (resp. lower) canonical basis of $V^{\otimes r}$ with the upper (resp. lower) canonical basis of $\mathscr{H}_r$ and well-known results about crystal lattices, where V is the natural representation of $U_q(\mathfrak{gl}_n)$. The results we need are similar to those in [14,11,7] and follow easily from results of [25,22]. Brundan's paper [7] is particularly well adapted to our needs and we follow it closely.