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Accessing the cohomology of discrete groups above their virtual cohomological dimension ☆

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A B S T R A C T

We introduce a method to explicitly determine the Farrell–Tate cohomology of discrete groups. We apply this method to the Coxeter triangle and tetrahedral groups as well as to the Bianchi groups, i.e. $\mathrm{PSL}_2(\mathcal{O})$ for $\mathcal{O}$ the ring of integers in an imaginary quadratic number field, and to their finite index subgroups. We show that the Farrell–Tate cohomology of the Bianchi groups is completely determined by the numbers of conjugacy classes of finite subgroups. In fact, our access to Farrell–Tate cohomology allows us to detach the information about it from geometric models for the Bianchi groups and to express it only with the group structure. Formulae for the numbers of conjugacy classes of finite subgroups have been determined in a thesis of Krämer, in terms of elementary number-theoretic information on $\mathcal{O}$. An evaluation of these formulae for a large number of Bianchi groups is provided numerically in the electronically released appendix to this paper. Our new insights about their homological torsion allow us to give a conceptual description of the cohomology ring structure of the Bianchi groups.

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1. Introduction

Our objects of study are discrete groups Γ such that Γ admits a torsion-free subgroup of finite index. By a theorem of Serre, all the torsion-free subgroups of finite index in Γ have the same cohomological dimension; this dimension is called the virtual cohomological dimension (abbreviated vcd) of Γ. Above the vcd, the (co)homology of a discrete group is determined by its system of finite subgroups. We are going to discuss it in terms of Farrell–Tate cohomology (which we will by now just call Farrell cohomology). The Farrell cohomology $\widehat{\mathrm{H}}^q$ is identical to group cohomology H^q in all degrees q above the vcd, and extends in lower degrees to a cohomology theory of the system of finite subgroups. Details are elaborated in [4, Chapter X]. So for instance considering the Coxeter groups, the virtual cohomological dimension of all of which vanishes, their Farrell cohomology is identical to all of their group cohomology. In Section 2, we will introduce a method of how to explicitly determine the Farrell cohomology: By reducing torsion sub-complexes. This method has also been implemented on the computer [6], which allows us to check the results that we obtain by our arguments. We apply our method to the Coxeter triangle and tetrahedral groups in Section 3, and to the Bianchi groups in Sections 4 through 6.

In detail, we require any discrete group Γ under our study to be provided with a cell complex on which it acts cellularly. We call this a *Γ-cell complex*. Let X be a Γ-cell complex; and let ℓ be a prime number. Denote by $X_{(\ell)}$ the set of all the cells σ of X, such that there exists an element of order ℓ in the stabilizer of the cell σ. In the case that the stabilizers are finite and fix their cells point-wise, the set $X_{(\ell)}$ is a Γ-sub-complex of X, and we call it the *ℓ-torsion sub-complex*.

For the Coxeter tetrahedral groups, generated by the reflections on the sides of a tetrahedron in hyperbolic 3-space, we obtain the following. Denote by $\mathcal{D}_\ell$ the dihedral group of order 2ℓ.

Corollary 1 *(Corollary to Theorem 12). Let Γ be a Coxeter tetrahedral group, and $\ell > 2$ be a prime number. Then there is an isomorphism $\mathrm{H}_q(\Gamma; \mathbb{Z}/\ell) \cong (\mathrm{H}_q(\mathcal{D}_\ell; \mathbb{Z}/\ell))^m$, with m the number of connected components of the orbit space of the ℓ-torsion sub-complex of the Davis complex of Γ.*

We specify the exponent m in the tables in Figs. 2 to 4.

Some individual procedures of our method have already been applied as ad hoc tricks by experts since [23], usually without providing a printed explanation of the tricks. An essential advantage of establishing a systematic method rather than using a set of ad hoc tricks, is that we can find ways to compute directly the quotient of the reduced torsion sub-complexes, working outside of the geometric model and skipping the often very laborious calculation of the orbit space of the Γ-cell complex. This provides access to the cohomology of many discrete groups for which the latter orbit space calculation is far out of reach. For instance, for the Bianchi groups, over a dozen of implementations