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Self similarity of dihedral tilings

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ABSTRACT

We show that for any prime number $n = 2r+1 \geqslant 5$ there exist r planar tilings with self-similar vertex set and the symmetry of a regular n-gon (D_n-symmetry). The tiles are the rhombi with angle $\pi k/n$ for $k = 1, \dots, r$.

1. Introduction

Tilings of euclidean plane with a dihedral (D_n-)symmetry for $n \neq 2, 3, 4, 6$ must be aperiodic, due to the crystallographic restriction: there is no translation preserving the tiling. However, there can be another type of ordering: self similarity. A tiling of the full plane $\mathbb{R}^2$ is called *self similar* if its vertex set V contains a subset V' which is a homothetic image of V, i.e. $V' = \lambda V$ for some $\lambda > 1$. It is our aim to show the following theorem:

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Theorem 1. *When $n \geqslant 5$ is a prime, there exist self similar planar tilings with D_n-symmetry.*

The case $n = 5$ consists of the two well known Penrose tilings with exact pentagon symmetry [3,1], while $n = 7, 11$ have been discussed in [2]. Pictures of the $n = 7$ tilings can be found in [2] and [4].

We construct the tilings using the projection method [1], see also [2]: Our tilings are obtained by orthogonal projection of a subset of the grid $\mathbb{Z}^n \subset \mathbb{R}^n$ onto a 2-dimensional affine subspace E; the projected subset is the intersection of $\mathbb{Z}^n$ with the so called *strip* $\Sigma_E = E + I^n$ with $I = (0,1)$. The vertex set of the tiling is $V_E = \pi_E(\mathbb{Z}^n \cap \Sigma_E)$, and the tiles are projections of unit squares in $\mathbb{R}^n$ all of whose vertices belong to $\mathbb{Z}^n \cap \Sigma_E$. This tiling is well defined provided that E is *in general position* with respect to $\mathbb{Z}^n$, i.e. for every point of E at most 2 coordinates can be integers [1,5]. Assigning the n vertices of an n-gon to the standard unit vectors $e_1, \ldots, e_n$ of $\mathbb{R}^n$, we obtain a linear action of the dihedral group D_n onto $\mathbb{R}^n$. If $n = 2r + 1$ is odd, this representation decomposes into a 1-dimensional fixed space $\mathbb{R}d$ with $d = \sum_i e_i$ and irreducible 2-dimensional subrepresentations $E_1, \ldots, E_r$. We will choose E parallel to E_1, say $E = E_1 + a$ for some $a \in \mathbb{R}^n$. This tiling will have local D_n symmetry at many places. But in order to have global D_n symmetry we will choose $a = \frac{k}{n}d$ where $1 \leqslant k \leqslant n-1$.

The self similarity will be caused by a self adjoint D_n-invariant integer matrix S ("inflation matrix") which is integer invertible on $W := d^\perp$ (i.e. there is another D_n-invariant symmetric integer matrix T with $ST = TS = I$ on W) and which has eigenvalues λ_i with $|\lambda_i| > 1$ on each 2-dimensional component E_i of W for $i \geqslant 2$. Then S acts as a contraction on E_1 and an expansion on the other E_i, and we have[1] $S(\Sigma) \supset \Sigma'$ where $\Sigma' = E' + I^n$ with $E' = S(E) = E_1 + Sa$. Projecting the grid points in Σ' onto E' yields the point set $V_{E'}$. By projecting the grid points in $S(\Sigma)$ onto E' we obtain a larger point set $S(V_E) \supset V_{E'}$. Since E_1 is an eigenspace of S, the set $S(V_E)$ is homothetic to V_E. When V_E is invariant under D_n, so is also $S(V_E)$ and $V_{E'}$. There are only finitely many of such tilings with full D_n-symmetry. Therefore, passing to a power of S if necessary, we can arrange for $V_{E'}$ and V_E to be homothetic. This reduces the proof of the theorem to the construction of such a matrix S.

The D_n-invariant tilings are not so special as it seems; in fact any tiling corresponding to $E_1 + a$ with $a \in d^\perp$ is almost isometric to any of the symmetric tilings, as will be explained in Theorem 2 below.

2. Dihedral tilings

Let D_n denote the group of all rotations and reflections of a regular n-gon (Dihedral group). It acts by certain permutations on the set of vertices of the n-gon which may

[1] More precisely, since S is not integer invertible on $\mathbb{R}d$, we might have to pass to a suitable power of S, see [2].