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Category equivalences involving graded modules over weighted path algebras and weighted monomial algebras

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ABSTRACT

Let k be a field, Q a finite directed graph, and kQ its path algebra. Make kQ an $\mathbb{N}$-graded algebra by assigning each arrow a positive degree. Let I be an ideal in kQ generated by a finite number of paths and write $A = kQ/I$. Let $\operatorname{QGr} A$ denote the quotient of the category of graded right A-modules modulo the Serre subcategory consisting of those graded modules that are the sum of their finite dimensional submodules. This paper shows there is a finite directed graph Q' with all its arrows placed in degree 1 and an equivalence of categories $\operatorname{QGr} A \equiv \operatorname{QGr} kQ'$. A result of Smith now implies that $\operatorname{QGr} A \equiv \operatorname{Mod} S$, the category of right modules over an ultramatricial, hence von Neumann regular, algebra S.

1.

1.1. Let k be a field and A an $\mathbb{N}$-graded k-algebra. Let $\operatorname{Gr} A$ be the category with objects the $\mathbb{Z}$-graded right A-modules and morphisms the degree preserving graded A-module homomorphisms. Let $\operatorname{Fdim} A \subseteq \operatorname{Gr} A$ be the localizing subcategory of modules

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that are the sum of their finite-dimensional submodules. Let $\mathrm{QGr}\, A$ denote the quotient of $\mathrm{Gr}\, A$ by $\mathrm{Fdim}\, A$ and

$$\pi^* : \mathrm{Gr}\, A \to \mathrm{QGr}\, A$$

the canonical quotient functor. As $\mathrm{Fdim}\, A$ is localizing, π^* has a right adjoint which we will denote by π_*.

For $M \in \mathrm{Gr}\, A$, let $M(1) \in \mathrm{Gr}\, A$ be M as a right module with grading given by $M(1)_i := M_{i+1}$. We call $M(1)$ the *shift* of M. Shifting determines an auto-equivalence $(1) : \mathrm{Gr}\, A \to \mathrm{Gr}\, A$. The shift functor descends to an auto-equivalence on $\mathrm{QGr}\, A$ which we still denote by (1).

1.2. Question

Let $F = k\langle x_1, \dots, x_n\rangle$ be the free algebra endowed with an $\mathbb{N}$-grading induced by fixing $\deg(x_i) \geqslant 1$ for all i. What does $\mathrm{QGr}\, F$ look like?

The answer when $\deg(x_i) = 1$ for all i is in [5].

1.3. This paper shows that $\mathrm{QGr}\, F \equiv \mathrm{QGr}\, kQ$ where kQ is the path algebra of a finite quiver Q, graded by giving each arrow degree 1, and defined as follows: view F as the path algebra of the weighted quiver with one vertex and n loops of degrees $\deg(x_i)$. If $\deg(x_i) > 1$, replace the loop for x_i by $\deg(x_i) - 1$ vertices and a cycle through them consisting of $\deg(x_i)$ arrows each of degree 1. The answer to Question 1.2, combined with a result in [6], says that $\mathrm{QGr}\, F \equiv \mathrm{Mod}\, S$ where S is an ultramatricial, hence von Neumann regular, algebra.

The question this paper answers is a little more general.

1.4. Consider the categories $\mathrm{QGr}\, A$ where A is a finitely presented $\mathbb{N}$-graded k-algebra belonging to one of the following five classes:

- **PA1:** Path algebras of finite quivers with grading induced by declaring that all arrows have degree 1; this implies that the degree of a path is equal to its length.
- **WPA:** Weighted path algebras of finite quivers—this is a path algebra with grading given by assigning each arrow a degree $\geqslant 1$.
- **MA:** Monomial algebras: these are algebras of the form kQ/I where kQ is a weighted path algebra of a finite quiver and I is an ideal generated by a finite set of paths.
- **CMA:** Connected monomial algebras: these are monomial algebras kQ/I in which Q has only one vertex.
- **CMA1:** Connected monomial algebras that are generated by elements of degree 1.

The following diagram depicts the inclusions between the five classes: each class is contained in the class “above” it.