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On some sets of generators of finite groups

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A B S T R A C T

In several papers finite groups with fixed cardinalities of sets of generators are investigated and sometimes are named $\mathcal{B}$-groups. Groups with all subgroups being $\mathcal{B}$-groups are called groups with the basis property.

In the first part of this paper we correct and generalize some characterizations of groups with the basis property. In the second part we consider groups satisfying analogous conditions, but only for sets of generators of prime power orders. The class of groups introduced here is much larger then the class of groups with the basis property. For example, it contains all nilpotent groups.

1. Preliminaries

All groups considered here are finite. For any group G let $\Phi(G)$ denote the Frattini subgroup of G. As in [4], groups with elements only of prime power order will be called *CP-groups*. For other notation, terminology and results on groups one can consult for example [5,14]. For needed information about fields, rings and modules see [9].

A subset X of a group G is said here to be:

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- *g-independent* if $\langle Y, \Phi(G)\rangle \neq \langle X, \Phi(G)\rangle$ for every $Y \subset X$;
- a *generating set* if $\langle X\rangle = G$ (or equivalently $\langle X, \Phi(G)\rangle = G$);
- a *g-base* of G, if X is a g-independent generating set of G.

Clearly, for every group G generating sets exist and every generating set contains a g-base of G. g-Bases of minimal and of maximal cardinalities are extensively studied, for example in [3,6,8,10,12,13,18]. We will continue these studies, especially those from [6,13,3]. Let us agree that:

- A group G has *property* $\mathcal{B}$ (is a $\mathcal{B}$-group) if every two g-bases of G have the same cardinality (span property in [1], property (M1) in [15]);
- A group G has the *basis property* if G and all its subgroups have property $\mathcal{B}$ [6,13,3].

$\mathcal{B}$-groups are well connected with Frattini subgroups (see [8,13]). We can summarize these informations in the following way

Proposition 1.1. *Let G be a group and let $H \leqslant \Phi(G) < G$ be a normal subgroup. Then G is a $\mathcal{B}$-group if and only if G/H is a $\mathcal{B}$-group.*

Due to the Burnside Basis Theorem we know that p-groups are $\mathcal{B}$-groups and even have the basis property. The following theorem summarizes some properties of groups with the basis property, proved in [6].

Theorem 1.2. *Let G be a group with the basis property. Then*:

1. *G is a CP-group*;
2. *G is soluble*;
3. *Every homomorphic image of G has the basis property*;
4. *If $G = G_1 \times G_2$ where G_i are nontrivial, then G is a p-group.*

In [3] claims (2) and (3) are generalized to all $\mathcal{B}$-groups.

2. Some examples

In characterizations of groups with the basis property, some extensions of p-groups by cyclic q-groups play an important role. We are going to provide a construction of such extensions from [13, §3] in a modified and generalized version. For this purpose let us formulate the following auxiliary result extracted from [9]:

Lemma 2.1. *Let $p \neq q$ be primes and let m be a nonnegative integer. Then there exists the smallest field $\mathbb{K} = \mathbb{K}(p, q, m)$ of characteristic p such that $\mathbb{K}$ contains all q^m-th roots of the unity. If $\rho_1, \ldots, \rho_s$ are all primitive q^m-th roots of 1 in $\mathbb{K}$, then*