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Weierstrass semigroups on double covers of genus 4 curves [☆]

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ABSTRACT

Let C be a complete non-singular irreducible curve of genus 4 over an algebraically closed field of characteristic 0. We determine all possible Weierstrass semigroups of ramification points on double covers of C which have genus greater than 11. Moreover, we construct double covers with ramification points whose Weierstrass semigroups are the possible ones.

1. Introduction

A submonoid H of the additive monoid $\mathbb{N}_0$ of non-negative integers is called a *numerical semigroup* if its complement $\mathbb{N}_0 \backslash H$ is a finite set. The cardinality of $\mathbb{N}_0 \backslash H$ is called the *genus* of H, which is denoted by $g(H)$. Let C be a complete non-singular irreducible

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curve over an algebraically closed field k of characteristic 0, which is called a *curve* in this paper. Let $k(C)$ be the field of rational functions on C. For a point P of C, we set

$$H(P) = \{\alpha \in \mathbb{N}_0 \mid \text{there exists } f \in k(C) \text{ with } (f)_\infty = \alpha P\},$$

which is called the *Weierstrass semigroup of* P. It is known that the Weierstrass semigroup of a point on a curve of genus g is a numerical semigroup of genus g.

Let $\pi : \tilde{C} \longrightarrow C$ be a double covering of a curve. We are interested in the Weierstrass semigroup $H(\tilde{P})$ of a ramification point $\tilde{P}$ on the double cover $\tilde{C}$ of C. A numerical semigroup is said to be *of double covering type*, if it is such a Weierstrass semigroup $H(\tilde{P})$. For a numerical semigroup $\tilde{H}$ we denote by $d_2(\tilde{H})$ the set consisting of the elements $\tilde{h}/2$ with even $\tilde{h} \in \tilde{H}$, which becomes a numerical semigroup. Using this notation we have $d_2(H(\tilde{P})) = H(\pi(\tilde{P}))$. Let $\tilde{g}$ be the genus of $\tilde{C}$. If C is the projective line, then $\tilde{P}$ is a Weierstrass point on the hyperelliptic curve $\tilde{C}$ for any $\tilde{g} \geqq 2$. Hence, the semigroup $H(\tilde{P})$ is generated by 2 and $2\tilde{g}+1$. If C is an elliptic curve, then the semigroup $H(\tilde{P})$ is either $\langle 3,4,5\rangle$ or $\langle 3,4\rangle$ or $\langle 4,5,6,7\rangle$ or $\langle 4,6,2\tilde{g}-3\rangle$ with $\tilde{g} \geqq 4$ or $\langle 4,6,2\tilde{g}-1,2\tilde{g}+1\rangle$ with $\tilde{g} \geqq 4$. Here for any positive integers $a_1, a_2, \ldots, a_n$ we denote by $\langle a_1, a_2, \ldots, a_n\rangle$ the additive monoid generated by $a_1, a_2, \ldots, a_n$. Conversely, there is a double cover of an elliptic curve with a ramification point whose Weierstrass semigroup is any semigroup of the above ones (for example, see [2,3]). Oliveira and Pimentel [8] studied Weierstrass semigroups $H(\tilde{P})$ in the case where the genus of C is 2. They showed that for a semigroup $\tilde{H} = \langle 6,8,10,n\rangle$ with an odd number $n \geqq 11$ there exists a double covering $\pi : \tilde{C} \longrightarrow C$ with a ramification point $\tilde{P}$ such that $H(\tilde{P}) = \tilde{H}$. Moreover, in [3] we showed that all numerical semigroups $\tilde{H}$ of genus $\geqq 6$ satisfying $g(d_2(\tilde{H})) = 2$ are of double covering type. In [5] we proved that every numerical semigroup $\tilde{H}$ of genus $\geqq 9$ with $g(d_2(\tilde{H})) = 3$ is of double covering type.

In this paper we will study the Weierstrass semigroups of ramification points on double covers of genus 4 curves. Namely, we prove

Main Theorem. *Let H be a numerical semigroup of genus 4 and $\tilde{H}$ a numerical semigroup with $d_2(\tilde{H}) = H$. If $g(\tilde{H}) \geqq 12$, then $\tilde{H}$ is of double covering type.*

But we note that the Main Theorem does not hold in the case where $g \geqq 5$. That is to say, for any $g \geqq 5$ there are numerical semigroups $\tilde{H}$ with $g(d_2(\tilde{H})) = g$ and $g(\tilde{H}) \geqq 3g$ which are not of double covering type (see [4]).

In Section 2 we will treat known facts and new results which work well for numerical semigroups of any genus with some properties. We know that a numerical semigroup of genus 4 is either $\langle 2,9\rangle$ or $\langle 3,5\rangle$ or $\langle 3,7,8\rangle$ or $\langle 4,5,6\rangle$ or $\langle 4,5,7\rangle$ or $\langle 4,6,7,9\rangle$ or $\langle 5,6,7,8,9\rangle$. By [7] any numerical semigroup $\tilde{H}$ with $d_2(\tilde{H}) = \langle 2,9\rangle$ is of double covering type. From Section 3 to Section 8 we will prove the Main Theorem for each numerical semigroup $\tilde{H}$ with $g(d_2(\tilde{H})) = 4$ and $d_2(\tilde{H}) \neq \langle 2,9\rangle$.