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The mirabolic Hecke algebra

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A R T I C L E I N F O

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A B S T R A C T

The Iwahori–Hecke algebra of the symmetric group is the convolution algebra of GL_n-invariant functions on the variety of pairs of complete flags over a finite field. Considering convolution on the space of triples of two flags and a vector we obtain the mirabolic Hecke algebra R_n, which had originally been described by Solomon. In this paper we give a new presentation for R_n, which shows that it is a quotient of a cyclotomic Hecke algebra as defined by Ariki and Koike. From this we recover the results of Siegel about the representations of R_n. We use Jucys–Murphy elements to describe the center of R_n and to give a $\mathfrak{gl}_\infty$-structure on the Grothendieck group of the category of its representations, giving 'mirabolic' analogues of classical results about the Iwahori–Hecke algebra. We also outline a strategy towards a proof of the conjecture that the mirabolic Hecke algebra is a cellular algebra.

1. Introduction

1.1. The Iwahori–Hecke algebra H_n of the symmetric group S_n is an example of a convolution algebra. The basic setting for convolution is the following: we have a finite

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set X and we take $E = \mathbb{C}(X \times X)$ to be the vector space of all complex valued functions on $X \times X$. Then given two functions f, g we define their convolution to be

$$(f * g)(x, y) = \sum_{z \in X} f(x, z) g(z, y). \tag{1}$$

This defines an associative product on E. If G is a group acting on X, then we have the diagonal action of G on $X \times X$ which induces a G-action on E. We can then consider the algebra $E^G \simeq \mathbb{C}(G \backslash X \times X)$ of functions that are invariant under the group action, with the same convolution product of (1). Let $G = \mathrm{GL}_n(\mathbb{F}_q)$, and B be the subgroup of upper triangular matrices. Then we can take $X = G/B$, the space of all complete flags in $\mathbb{F}_q^n$. From the Bruhat decomposition, it follows that the G-orbits on $G/B \times G/B$ are parametrized by the symmetric group S_n. As we will discuss in Section 2, for $X = G/B$, the resulting convolution algebra is the Iwahori–Hecke algebra of S_n.

The action of G on the space $G/B \times G/B \times \mathbb{F}_q^n$ still has finitely many orbits. This follows from the results of [21] but it is also a special case of the work of Magyar, Weyman and Zelevinsky, which in [15] have classified all cases in which the G action on triples of flags has finitely many orbits. We can then define a convolution product on the space of invariant functions $\mathbb{C}(G/B \times G/B \times \mathbb{F}_q^n)^G$. We call the resulting algebra R_n the *mirabolic Hecke algebra*.

1.2. For a field $\Bbbk$, the action of $\mathrm{GL}_n(\Bbbk)$ on varieties of flags and pairs of flags is a classical topic of study. Recently, generalizations of these constructions have been appearing, including the extra data of a vector or a line. This is what we mean by the 'mirabolic' setting. The name comes from the mirabolic subgroup $P \subset \mathrm{GL}_n(\Bbbk)$, which is the subgroup that fixes a nonzero vector in $V = \Bbbk^n$. This is because in general, for a G-variety X, the P-orbits on X are obviously in a 1–1 correspondence with G-orbits on $X \times (V \setminus \{0\})$.

One case in which such a generalization arises is the following. If G/B is the variety of complete flags in V, then it is interesting to study the action of G on $G/B \times G/B \times V$. One reason why this is important is because $\mathfrak{D}$-modules on $G/B \times G/B \times V$ are closely related to mirabolic character $\mathfrak{D}$-modules. These are certain $\mathfrak{D}$-modules on $G \times V$, which arise when studying the spherical trigonometric Cherednik algebra, see [8].

Another example is the work of Achar and Henderson [1] extending the nilpotent cone $\mathcal{N} \subset \mathrm{End}(V)$ to the 'enhanced nilpotent cone' $\mathcal{N} \times V$. Here the group G acts on V in the obvious way and on $\mathcal{N}$ by conjugation. Their motivation was the work of S. Kato [14], which had introduced the 'exotic nilpotent cone', of which $\mathcal{N} \times V$ is a simplification. Kato uses the exotic nilpotent cone to establish an 'exotic' Springer correspondence and to give a geometric construction of the affine Hecke algebra of type $C_n^{(1)}$.

1.3. The focus of the present paper is the 'mirabolic' Hecke algebra R_n. This had originally been defined, in different terms, by Solomon in [21], and its irreducible