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The closed cone of a rational series is rational polyhedral ☆

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A B S T R A C T

For a multivariate power series f, let $\mathrm{Cone}(f)$ denote the cone generated by the exponents of the monomials with nonzero coefficients. Assume that f is an expansion of a rational function p/q with $\gcd(p,q) = 1$. Then we prove that the closure $\overline{\mathrm{Cone}}(f)$ is equal to $\mathrm{Cone}(p) + \mathrm{Cone}(q)$. As applications, we show the irrationality of Euler–Chow series of certain algebraic varieties.

1. Introduction

In various branches of mathematics naturally arise multivariate generating series, and sometimes they are rational.

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The Euler–Chow series $EC_{X,p}$ defined by J. Elizondo is one such example. Here X is an algebraic variety and p is a non-negative integer, and $EC_{X,p}$ is defined as the generating series of the Euler characteristics of Chow varieties of X parametrizing p-dimensional effective cycles. Roughly speaking, it can be considered as a power series in n variables, where n is the rank of $H_{2p}(X,\mathbb{Z})$. Elizondo [4] proved that $EC_{X,p}$ is rational for any simplicial toric variety X, and asked whether it is rational for an arbitrary variety X or not. He proposed the following possible counterexample: Let X be the blow-up of $\mathbb{P}^2$ at 9 points in general position. Then the pseudo-effective cone of X is not polyhedral, and so it seems unlikely that the Euler–Chow series $EC_{X,1}(t)=\sum_{D\in \mathrm{Pic}(X)} h^0(D)t^D$ could be rational.

In this paper, we show that this reasoning does work. Let k be a field. For a power series $f=\sum a_{i_1,\dots,i_n}x_1^{i_1}\cdots x_n^{i_n}\in k[\![x_1,\dots,x_n]\!]$, let

$$\mathrm{Supp}(f):=\{(i_1,\dots,i_n)\mid a_{i_1,\dots,i_n}\neq 0\}\subset \mathbb{Z}^n$$

and

$$\mathrm{Cone}(f):=\sum_{\mathbf{m}\in \mathrm{Supp}(f)} \mathbb{R}_{\geqslant 0}\mathbf{m}\subset \mathbb{R}^n.$$

Then we show that the closure $\overline{\mathrm{Cone}}(f)$ is rational polyhedral when f is a rational function. More precisely, the following holds.

Theorem 1.1. *Assume that there exists a nonzero polynomial $q\in k[x_1,\dots,x_n]$ such that $p:=qf$ belongs to $k[x_1,\dots,x_n]$. Choose such a polynomial q with the property that* $\gcd(p,q)=1$. *Then $q(0)\neq 0$,*

$$f=\frac{p}{q(0)}\sum_{i=0}^{\infty}\left(-\frac{q-q(0)}{q(0)}\right)^i \tag{1.1}$$

and $\overline{\mathrm{Cone}}(f)$ is equal to $\mathrm{Cone}(p)+\mathrm{Cone}(q)$.

This assertion is so natural that one would expect it to be already known, but we could not find it stated anywhere.

Remark 1.2. Let us list a few related directions of works.

(1) In the context of tropical geometry, [3] studied the relation between the Newton polytope of a polynomial q, the tropical variety associated to q and the set of different expansions of $1/q$. More generally, Puiseux expansions of an algebraic function f are studied in [7], for example. See Remark 2.1 for a little more detail.

(2) For an algebraic power series over a finite field, the set of exponents with vanishing coefficients form a "p-automatic set." See [1].