Journal of Algebra 395 (2013) 12–23

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Strongly liftable schemes and the Kawamata–Viehweg vanishing in positive characteristic III ☆

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A R T I C L E I N F O

Article history:
Received 30 January 2013
Available online 29 August 2013
Communicated by Steven Dale Cutkosky

Dedicated to Professor Yujiro Kawamata for his sixtieth birthday

Keywords:
Strongly liftable scheme
The Kawamata–Viehweg vanishing
Positive characteristic

A B S T R A C T

A smooth scheme X over a field k of positive characteristic is said to be strongly liftable over $W_2(k)$, if X and all prime divisors on X can be lifted simultaneously over $W_2(k)$. In this paper, we first deduce the Kummer covering trick over $W_2(k)$, which can be used to construct a large class of smooth projective varieties liftable over $W_2(k)$, and to give a direct proof of the Kawamata–Viehweg vanishing theorem on strongly liftable schemes. Secondly, we generalize almost all of the results in [18,19] to the case where everything is considered over $W(k)$, the ring of Witt vectors of k.

1. Introduction

Throughout this paper, we always work over *an algebraically closed field k of characteristic $p > 0$* unless otherwise stated. A smooth scheme X is said to be strongly liftable over $W_2(k)$, if X and all prime divisors on X can be lifted simultaneously over $W_2(k)$. This notion was first introduced in [18] to study the Kawamata–Viehweg vanishing theorem in positive characteristic, furthermore, some examples and properties of strongly liftable schemes were also given in [18,19].

In this paper, we shall continue to study strongly liftable schemes. First of all, we deduce the Kummer covering trick over $W_2(k)$ by means of the logarithmic techniques developed by K. Fujiwara, K. Kato and C. Nakayama. The Kummer covering trick over $W_2(k)$ can be used to construct a large class of smooth projective varieties liftable over $W_2(k)$ (see Section 3 for more details).

☆ This paper was partially supported by the National Natural Science Foundation of China (Grant Nos. 11231003 and 11271070), and the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry.

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0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2013.07.027

Theorem 1.1. *Let X be a smooth projective variety strongly liftable over $W_2(k)$, D a $\mathbb{Q}$-divisor on X such that the fractional part $\langle D\rangle = \sum_{i\in I} \frac{a_i}{b_i} D_i$ satisfies that $0 < a_i < b_i$, $(a_i, b_i) = 1$ and $p \nmid b_i$ for all $i \in I$, and $\sum_{i\in I} D_i$ is simple normal crossing. Then there exists a finite Galois morphism $\tau : Y \to X$ with Galois group $G = \mathrm{Gal}(K(Y)/K(X))$, such that $\tau^* D$ is integral and Y is a smooth projective variety liftable over $W_2(k)$.*

As an application, the Kummer covering trick over $W_2(k)$ gives a direct proof of the Kawamata–Viehweg vanishing theorem on strongly liftable schemes.

Corollary 1.2. *Let X be a strongly liftable smooth projective variety of dimension d, and D an ample $\mathbb{Q}$-divisor on X such that $\mathrm{Supp}\langle D\rangle$ is simple normal crossing. Then we have $H^i(X, K_X + \ulcorner D \urcorner) = 0$ for any $i > d - \inf(d, p)$.*

Secondly, we generalize almost all of the results in [18,19] to the case where everything is considered over $W(k)$, the ring of Witt vectors of k. A smooth scheme X is said to be strongly liftable over $W(k)$, if X and all prime divisors on X can be lifted simultaneously over $W(k)$. The following is the main result in Section 4.

Theorem 1.3. *The following varieties are strongly liftable over $W(k)$:*

(i) $\mathbb{A}^n_k$, $\mathbb{P}^n_k$ *and a smooth projective curve*;
(ii) *A smooth projective variety of Picard number* 1 *which is a complete intersection in* $\mathbb{P}^n_k$;
(iii) *A smooth projective rational surface*;
(iv) *A smooth projective toric variety.*

Finally, by means of cyclic covers over toric varieties, we can obtain a large class of smooth projective varieties liftable over $W(k)$ (see Theorem 4.12 and Corollary 4.13 for more details).

Corollary 1.4. *Let X be a smooth projective toric variety, and $\mathcal{L}$ an invertible sheaf on X. Let N be a positive integer prime to p, and D an effective divisor on X with $\mathcal{L}^N = \mathcal{O}_X(D)$ and $\mathrm{Sing}(D_{\mathrm{red}}) = \emptyset$. Let $\pi : Y \to X$ be the cyclic cover obtained by taking the N-th root out of D. Then Y is a smooth projective scheme liftable over $W(k)$.*

In Section 2, we will recall some definitions and preliminary results of liftings over $W_2(k)$. The Kummer covering trick over $W_2(k)$ will be treated in Section 3. We will give the generalizations to $W(k)$ in Section 4. For the necessary notions and results in birational geometry, we refer the reader to [11].

Notation. We use $[B] = \sum [b_i]B_i$ (resp. $\ulcorner B \urcorner = \sum \ulcorner b_i \urcorner B_i$, $\langle B\rangle = \sum \langle b_i\rangle B_i$) to denote the round-down (resp. round-up, fractional part) of a $\mathbb{Q}$-divisor $B = \sum b_i B_i$, where for a real number b, $[b] := \max\{n \in \mathbb{Z} \mid n \leqslant b\}$, $\ulcorner b \urcorner := -[-b]$ and $\langle b\rangle := b - [b]$. We use $\mathrm{Sing}(D_{\mathrm{red}})$ (resp. $\mathrm{Supp}(D)$) to denote the singular locus of the reduced part (resp. the support) of a divisor D. We use $K(X)$ to denote the field of rational functions of an integral scheme X.

2. Preliminaries

Definition 2.1. Let $W_2(k)$ be the ring of Witt vectors of length two of k. Then $W_2(k)$ is flat over $\mathbb{Z}/p^2\mathbb{Z}$, and $W_2(k) \otimes_{\mathbb{Z}/p^2\mathbb{Z}} \mathbb{F}_p = k$. The following definition [2, Definition 8.11] generalizes the definition [1, 1.6] of liftings of k-schemes over $W_2(k)$.

Let X be a noetherian scheme over k, and $D = \sum D_i$ a reduced Cartier divisor on X. A lifting of (X, D) over $W_2(k)$ consists of a scheme $\tilde{X}$ and closed subschemes $\tilde{D}_i \subset \tilde{X}$, all defined and flat over $W_2(k)$ such that $X = \tilde{X} \times_{\mathrm{Spec}\, W_2(k)} \mathrm{Spec}\, k$ and $D_i = \tilde{D}_i \times_{\mathrm{Spec}\, W_2(k)} \mathrm{Spec}\, k$. We write $\tilde{D} = \sum \tilde{D}_i$ and say that $(\tilde{X}, \tilde{D})$ is a lifting of (X, D) over $W_2(k)$, if no confusion is likely.