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On the involution module of $\mathrm{GL}_n(2^f)$

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ABSTRACT

For any group G the set of involutions $\mathcal{J}$ in G, that is, the set of group elements that have order two, forms a G-set under conjugation. The corresponding kG-permutation module $k\mathcal{J}$ is the involution module of G. Here k is an algebraically closed field of characteristic two. In this paper we discuss aspects of the involution module of the general linear group $\mathrm{GL}_n(2^f)$. We determine almost all components of this module. Furthermore we present a vertex and the Green correspondent of each component.

1. Introduction

The goal of this paper is to investigate the involution module of the general linear group $\mathrm{GL}_n(2^f)$, where $f \geqslant 1$ is an integer. We start with an introduction to the idea of the involution module. Also in this section we develop the necessary notation and state some important results which we employ in our work.

In Section 2 we present a partial decomposition of the involution module of $\mathrm{GL}_n(2^f)$. In Theorem 3.1 the possible vertices of a component of our involution module are given. By component we mean an indecomposable summand. Sections 4–8 focus on each of those possible candidates. Finally we summarize our results in Theorem 9.1, followed by some further observations.

1.1. The involution module

Let G be a finite group and let k be an algebraically closed field of characteristic 2. By $\mathcal{J}$ we denote the set of involutions in G, that is, the set of elements in G of order two. Then G acts on $\mathcal{J}$ by conjugation. In particular we obtain the kG-permutation module $k\mathcal{J}$. This module is called the *involution module* of G. In the paper [15], G.R. Robinson investigated the projective components of

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this module using the Frobenius–Schur indicator. Later J. Murray studied the involution module in general in [8–11]. He, too, worked with the Frobenius–Schur indicator, but also used block-theoretical methods, such as the defect class of a block. Furthermore P. Collings studied parts of the involution module of the symmetric group in his PhD thesis [4], focusing on the fixed point free involutions in the symmetric group $\mathrm{Sym}(n)$. Finally the author studied the involution module of the special linear group $\mathrm{SL}_2(2^f)$ in [12].

The main motivation of this paper is to study the involution module of the general linear group $\mathrm{GL}_n(2^f)$. We are able to determine the number of components and describe the vertex and Green correspondent of each component. However as many of our calculations are still valid for any prime number p, we present most results for $\mathrm{GL}_n(p^f)$ and k of characteristic p.

1.2. Notation

Throughout this paper let k be an algebraically closed field of prime characteristic p. Also let $f, n \geqslant 1$ be integers and set $q := p^f$. By $\mathbb{F}_q$ we mean the finite field with q elements. Our main group of interest is the general linear group $\mathrm{GL}_n(q)$, that is, the group of all invertible $n \times n$-matrices with entries in $\mathbb{F}_q$. For convenience we denote this group by GL_n in the following. Note that both the upper-triangular matrices in GL_n and the upper-triangular matrices in GL_n with ones on the main diagonal form subgroups of GL_n. We denote them by B_n and U_n, respectively. Furthermore U_n is a Sylow-p-subgroup of both B_n and GL_n. As is standard I_n denotes the identity matrix in GL_n. For integers $1 \leqslant k < l \leqslant n$ let $E_{k,l}(\alpha)$ be the $n \times n$-matrix with zeros everywhere, except for the (k,l)-entry which is $\alpha \in \mathbb{F}_q$. Then $F_{k,l} := \{I_n + E_{k,l}(\alpha)\colon\ \alpha \in \mathbb{F}_q\}$ is the subgroup of U_n of matrices where all entries off the main diagonal are zero, except for the (k,l)-entry which can be anything in $\mathbb{F}_q$. Finally for any two integers $r, s \geqslant 0$ we define $\mathrm{GL}_{r,s} := \mathrm{GL}_r \times \mathrm{GL}_s$, where GL_0 is the trivial group.

Next let $\lambda_1, \dots, \lambda_t \geqslant 0$ such that $\lambda_1 + \dots + \lambda_t = n$. Then if $A_r \in \mathrm{GL}_{\lambda_r}$, for $r = 1, \dots, t$, we define $D_n(A_1, A_2, \dots, A_t)$ as that matrix in GL_n that has the matrices $A_1, \dots, A_t$ on its main diagonal and zeros everywhere else. Likewise $D_n(A_1\bullet, A_2\bullet, \dots, A_t)$ denotes a matrix with the matrices $A_1, \dots, A_t$ on its main diagonal, zeros below and arbitrary elements in $\mathbb{F}_q$ above. Note that $D_n(A_1\bullet, A_2\bullet, \dots, A_t)$ is not unique, but in our considerations it does not matter what the specific entries above the diagonal of matrices $A_1, \dots, A_t$ are. In the same sense we define the groups $D_n(H_1, H_2, \dots, H_t)$ and $D_n(H_1\bullet, H_2\bullet, \dots, H_t)$, for $H_r \leqslant \mathrm{GL}_{\lambda_r}$. Finally set $\mathrm{GP}_{\lambda_1,\dots,\lambda_r} := D_n(\mathrm{GL}_{\lambda_1}\bullet, \mathrm{GL}_{\lambda_2}\bullet, \dots, \mathrm{GL}_{\lambda_r})$. Note that $\mathrm{GP}_{\lambda_1,\dots,\lambda_r}$ is known as a parabolic subgroup of GL_n.

Still let $n \geqslant 1$. Then $\mathcal{W}_n$ denotes the group of permutation matrices in GL_n. Note that we can identify a permutation matrix with a unique permutation in the symmetric group $\mathrm{Sym}(n)$. In fact $\omega \in \mathrm{Sym}(n)$ corresponds to the permutation matrix $(\delta_{k,\omega(l)})_{k,l}$, where $\delta_{-,-}$ denotes the Kronecker-symbol.

Finally we discuss some block theory of GL_n. Refer to [6] and [16] for definitions and more details. A block of GL_n has either full defect or is of defect zero. There are exactly $q-1$ of each type. Also the module $k_{B_n}\uparrow^{\mathrm{GL}_n}$ has a unique irreducible projective component St_n, which is called the *Steinberg module*. This module is self-dual and has dimension $q^{\binom{n}{2}}$. Furthermore the center $\mathrm{Z}(\mathrm{GL}_n)$ of GL_n acts trivially on $k_{B_n}\uparrow^{\mathrm{GL}_n}$. This follows since $\mathrm{Z}(\mathrm{GL}_n) = \{\alpha I_n\colon\ \alpha \in \mathbb{F}_q^*\}$ is a normal subgroup of B_n. In particular $\mathrm{Z}(\mathrm{GL}_n)$ acts trivially on St_n.

Let $\mathcal{S}$ denote the GL_n-representation corresponding to St_n. For $A \in \mathrm{GL}_n$ and $j = 0, 1, \dots, q-2$, we define $\mathcal{S}^j(A) := (\det(A))^j \cdot \mathcal{S}(A)$. Then $\mathcal{S}^j$ is a projective irreducible GL_n-representation. We denote the corresponding GL_n-module by St_n^j. If $(\mathrm{St}_n^j)^*$ denotes the dual of St_n^j, then $(\mathrm{St}_n^j)^* = \mathrm{St}_n^{q-1-j}$. The modules $\mathrm{St}_n, \mathrm{St}_n^1, \dots, \mathrm{St}_n^{q-2}$ are all the projective irreducible GL_n-modules. As every block of defect zero contains a unique irreducible projective module we let B_j^z denote the block that contains St_n^j, for $j = 0, 1, \dots, q-2$. Thus

$$B_j^z = \mathrm{St}_n^j \otimes \mathrm{St}_n^{q-1-j}, \quad \text{as } \mathrm{GL}_{n,n}\text{-modules}. \tag{1}$$

In particular B_j^z is projective and irreducible as a $\mathrm{GL}_{n,n}$-module.