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Representations of the Witt superalgebra $W(2)$

Feifei Duan [a,*], Bin Shu [b,1]

[a] Department of Mathematics, Zhejiang University (Yuquan Campus), Hangzhou, Zhejiang 310027, PR China
[b] Department of Mathematics, East China Normal University (Minhang Campus), Shanghai 200241, PR China

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ABSTRACT

In this paper, we study representations of the Lie superalgebra $W(2)$ of Witt type over an algebraically closed field of characteristic $p > 2$. All irreducible modules and their dimensions are determined, up to isomorphisms. Thus, all irreducible modules and their dimensions of $\mathfrak{osp}(2|2)$ are determined in prime characteristic, up to isomorphisms.

1. Introduction

This paper is somewhat a sequel to [4] and [5]. In the latter articles, the authors studied restricted representations for the Witt superalgebras $W(n)$ and non-restricted representations for the generalized Witt superalgebras $W(m:n:\mathbf{1})$ respectively. Character formulas for irreducible restricted modules over $W(n)$ were obtained in [4], while partial results on the description of non-restricted irreducible modules over $W(m:n:\mathbf{1})$ were presented in [5].

Based on the above papers, we continue to study the representations of $W(2)$ over an algebraically closed filed $\mathbf{k}$ of characteristic $p > 2$. In this special case, the purely-even subalgebra is isomorphic to $\mathfrak{gl}(2)$, for which the irreducible modules can be described precisely. So we can manage to give a full description of all its non-restricted irreducible modules of $W(2)$, as well as give some precise information on submodules of Kac modules in the restricted module category over $W(2)$. The structure of

* Corresponding author.
E-mail addresses: duanfeifei0918@126.com (F. Duan), bshu@math.ecnu.edu.cn (B. Shu).
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this paper is as follows. In Section 2, we introduce some basic notions and definitions for the Witt Lie superalgebras, as well as some known results which will be used later in the paper. In Section 3, the information on modular representations of $\mathfrak{gl}(2)$, which is particularly important, is represented fully for our later arguments. In Section 4, we deal with restricted representations of $W(2)$, under which case all submodules of the Kac module are described. In Section 5, we focus on non-restricted representations. According to Remark 3.1, we only need to study irreducible modules of the semisimple p-characters and of nilpotent p-characters. Here p-characters are defined on the even parts of the Lie superalgebras, which are isomorphic to $\mathfrak{gl}(2)$. The readers can be referred to [1] and [2] for the meaning of semisimple p-characters and nilpotent p-characters. In both of the semisimple and nilpotent cases, we obtain a criterion on the simplicity of the Kac module $K_\chi(\lambda)$. We then present composition factors of $K_\chi(\lambda)$ whenever it's not irreducible. In addition, the bases of all the submodules are given. Consequently, the dimension of the irreducible quotient modules are given.

Recall that Weiqiang Wang and Lei Zhao gave in [6] a complete description of irreducible representations of classical Lie superalgebra $\mathfrak{osp}(1|2)$ in prime characteristic. In view that $W(2)$ is isomorphic to $\mathfrak{osp}(2|2)$ as Lie superalgebras, we actually present the complete information on irreducible modules of $\mathfrak{osp}(2|2)$ for every prime $p > 2$.

2. Preliminaries

Throughout the paper, $\mathbf{k}$ is assumed to be an algebraically closed field of characteristic $p > 2$. All vector (super)spaces are finite-dimensional ones over $\mathbf{k}$. A superspace $V = V_{\bar{0}} + V_{\bar{1}}$ is a vector space with $\mathbb{Z}_2$-grading, and the homogeneous element $v \in V_i$ for $i \in \mathbb{Z}_2$ has degree i (denoted by $\mathrm{d}(v)$). For simplicity, we will directly say vector spaces (resp. algebras, ideals, etc.), instead of saying superspaces (resp. superalgebras, superideals, etc.) if the context is clear.

2.1. Witt superalgebras

2.1.1. We will follow the references [3, §3.1] and [4] for the notions and notations of Witt superalgebras. For the convenience of the readers, we summarize some information on those superalgebras as below.

Let $\Lambda(n)$ be the Grassmann superalgebra over $\mathbf{k}$ with the generators $\xi_1, \xi_2, \ldots, \xi_n$. Denote by $\boldsymbol{W(n)}$ the superderivation algebra of the $\Lambda(n)$, which can be expressed as below

$$W(n) = \left\{ \sum_{i=1}^{n} f_i D_i \;\middle|\; f_i \in \Lambda(n) \right\}$$

where D_i is a superderivation defined via $D_i(\xi_j) = \delta_{ij},\ \forall i, j \in \{1, 2, \ldots, n\}$.

The superstructure on $W(n)$ arises from the $\mathbb{Z}$-grading over $W(n) = \bigoplus_{i=-1}^{n-1} W(n)_i$ where $W(n)_i = \mathbf{k}\text{-span}\{\xi_{t_1} \cdots \xi_{t_{i+1}} D_s \mid 1 \leqslant t_1 < \cdots < t_{i+1} \leqslant n,\ 1 \leqslant s \leqslant n\}$, with convention $\deg(\xi_i) = 1$ and $\deg(D_i) = -1$. More precisely, $W(n) = W(n)_{\bar{0}} + W(n)_{\bar{1}}$, where $W(n)_{\bar{0}} = \sum_{\bar{i}=\bar{0}} W(n)_i$, and $W(n)_{\bar{1}} = \sum_{\bar{i}=\bar{1}} W(n)_i$. Set $W(n)^{(i)} = \bigoplus_{j \geqslant i} W(n)_j$, $i = -1, 0, 1, \ldots, n-1$, there is a natural filtered structure on $W(n)$:

$$W(n) = W(n)^{(-1)} \supseteq W(n)^{(0)} \supseteq \cdots \supseteq W(n)^{(n-1)} \supseteq W(n)^{(n)} = 0.$$

The bracket $[\,,]$ on $W(n)$ is defined via

$$[f D_i, g D_j] = f D_i(g) D_j - (-1)^{\mathrm{d}(f D_i)\mathrm{d}(g D_j)} g D_j(f) D_i$$

for $f D_i, g D_j$ which are the homogeneous elements of $W(n)$.

Recall that $W(n)$ is a restricted Lie superalgebra, this is to say, $W(n)_{\bar{0}}$ is a restricted Lie algebra with p-mapping $[p]$ which is actually the usual pth power of derivations, and $W(n)_{\bar{1}}$ is a restricted