Journal of Algebra 391 (2013) 93–113

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Connected Hopf algebras of dimension p^2

Xingting Wang [1]

Department of Mathematics, University of Washington, Seattle, WA 98195, United States

ARTICLE INFO

Article history:
Received 10 August 2012
Available online 29 June 2013
Communicated by Nicolás Andruskiewitsch

MSC:
16T05
16S30

Keywords:
Hopf algebras
Lie algebras
Connected
Coradically graded
Cocommutative

ABSTRACT

Let H be a finite-dimensional connected Hopf algebra over an algebraically closed field $\mathbf{k}$ of characteristic $p > 0$. We provide the algebra structure of the associated graded Hopf algebra $\operatorname{gr} H$. Then, we study the case when H is generated by a Hopf subalgebra K and another element and the case when H is cocommutative. When H is a restricted universal enveloping algebra, we give a specific basis for the second term of the Hochschild cohomology of the coalgebra H with coefficients in the trivial H-bicomodule $\mathbf{k}$. Finally, we classify all connected Hopf algebras of dimension p^2 over $\mathbf{k}$.

1. Introduction

Let $\mathbf{k}$ denote a base field, algebraically closed of characteristic $p > 0$. In [5], all graded cocommutative connected Hopf algebras of dimension less than or equal to p^3 are classified by using W.M. Singer's theory of extensions of connected Hopf algebras [13]. In this paper, we classify all connected Hopf algebras of dimension p^2 over $\mathbf{k}$. We use the theories of restricted Lie algebras and Hochschild cohomology of coalgebras for restricted universal enveloping algebras.

Let H denote a finite-dimensional connected Hopf algebra in the sense of [9, Def. 5.1.5] with primitive space $\mathrm{P}(H)$, and K be a Hopf subalgebra of H. In Section 2, basic definitions related to and properties of H are briefly reviewed. In particular, we describe a few concepts concerning the inclusion $K \subseteq H$. We say that the *p-index* of K in H is $n - m$ if $\dim K = p^m$ and $\dim H = p^n$. The notion of the *first order* of the inclusion and a *level-one* inclusion are also given in Definition 2.3.

E-mail address: xingting@uw.edu.

[1] Research of the author was partially supported by the US National Science Foundation.

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2013.06.008

In Section 3, the algebra structure of a finite-dimensional connected coradically graded Hopf algebra is obtained (Theorem 3.1) based on a result for algebras representing finite connected group schemes over $\mathbf{k}$. It implies that the associated graded Hopf algebra $\operatorname{gr} H$ is isomorphic to as algebras

$$\mathbf{k}[x_1, x_2, \ldots, x_d]/\left(x_1^p, x_2^p, \ldots, x_d^p\right)$$

for some $d \geqslant 0$.

Section 4 concerns a simple case when H is generated by K and another element x. Suppose the p-index of K in H is d. Under an additional assumption, the basis of H as a left K-module is given in terms of the powers of x (Theorem 4.5). Moreover, if K is normal in H [9, Def. 3.4.1], then x satisfies a polynomial equation as follows:

$$x^{p^d} + \sum_{i=0}^{d-1} a_i x^{p^i} + b = 0$$

for some $a_i \in \mathbf{k}$ and $b \in K$.

Section 5 deals with the special case when H is cocommutative. It is proved in Proposition 5.2 that such Hopf algebra H is equipped with a series of normal Hopf subalgebras $\mathbf{k} = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_n = H$ satisfying certain properties. If we apply these properties to the case when $\mathrm{P}(H)$ is one-dimensional, then we have N_1 is generated by $\mathrm{P}(H)$ and each N_i has p-index one in N_{i+1} (Corollary 5.3). In Theorem 5.4, we give locality criterion for H in terms of its primitive elements. This result, after dualization, is equivalent to a criteria for unipotency of finite connected group schemes over $\mathbf{k}$, as shown in Remark 5.5.

In Section 6, we take the Hopf subalgebra $K = u(\mathfrak{g})$, the restricted universal enveloping algebra of some finite-dimensional restricted Lie algebra $\mathfrak{g}$. We consider the Hochschild cohomology of the coalgebra K with coefficients in the trivial bicomodule $\mathbf{k}$, namely $\mathrm{H}^\bullet(\mathbf{k}, K)$. Then the Hochschild cohomology can be computed as the homology of the cobar construction of K. In Proposition 6.2, we give a specific basis for $\mathrm{H}^2(\mathbf{k}, K)$. We further show, in Lemma 6.5, that $\bigoplus_{n\geqslant 0} \mathrm{H}^n(\mathbf{k}, K)$ is a graded restricted $\mathfrak{g}$-module via the adjoint map. When the inclusion $K \subseteq H$ has first order $n \geqslant 2$, the differential d^1 in the cobar construction of H induces a restricted $\mathfrak{g}$-module map from H_n into $\mathrm{H}^2(\mathbf{k}, K)$, whose kernel is K_n (Theorem 6.6). Concluded in Theorem 6.7, if $K \neq H$, we can find some $x \in H \setminus K$ with the following comultiplication

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \omega\left(\sum_i \alpha_i x_i\right) + \sum_{j<k} \alpha_{jk} x_j \otimes x_k$$

where $\{x_i\}$ is a basis for $\mathfrak{g}$.

Finally, the classification of connected Hopf algebras of dimension p^2 over $\mathbf{k}$ is accomplished in Section 7. Assume $\dim H = p^2$. We apply results on H from previous sections, i.e., Corollary 5.3 and Theorem 6.7. The main result is stated in Theorem 7.4 and divided into two cases. When $\dim \mathrm{P}(H) = 2$, based on the classification of two-dimensional Lie algebras with restricted maps (see Appendix A), there are five non-isomorphic classes

(1) $\mathbf{k}[x, y]/(x^p, y^p)$,
(2) $\mathbf{k}[x, y]/(x^p - x, y^p)$,
(3) $\mathbf{k}[x, y]/(x^p - y, y^p)$,
(4) $\mathbf{k}[x, y]/(x^p - x, y^p - y)$,
(5) $\mathbf{k}\langle x, y\rangle/([x, y] - y, x^p - x, y^p)$,

where x, y are primitive. When $\dim \mathrm{P}(H) = 1$, H must be commutative and there are three non-isomorphic classes