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Finite groups whose prime graphs are regular

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ABSTRACT

Let G be a finite group and let $\mathrm{Irr}(G)$ be the set of all irreducible complex characters of G. Let $\mathrm{cd}(G)$ be the set of all character degrees of G and denote by $\rho(G)$ the set of primes which divide some character degrees of G. The prime graph $\Delta(G)$ associated to G is a graph whose vertex set is $\rho(G)$ and there is an edge between two distinct primes p and q if and only if the product pq divides some character degree of G. In this paper, we show that the prime graph $\Delta(G)$ of a finite group G is 3-regular if and only if it is a complete graph with four vertices.

1. Introduction

Given a finite group G, let $\mathrm{Irr}(G)$ be the set of all irreducible complex characters of G and let $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ be the set of character degrees of G. The set of primes which divide some character degrees of G is denoted by $\rho(G)$. The *prime graph* $\Delta(G)$ associated to G is a graph whose vertex set is $\rho(G)$ and there is an edge between two distinct primes p and q in $\rho(G)$ if and only if the product pq divides some character degree $a \in \mathrm{cd}(G)$. The prime graph $\Delta(G)$ of a finite group G is a useful tool in studying the character degree set $\mathrm{cd}(G)$. This graph has been studied extensively over the last 20 years. We refer the readers to a recent survey by M. Lewis [9] for results concerning this graph and related topics.

In this paper, we are going to study the following question: Which graphs can occur as the prime graphs of finite groups? This is one of the basic questions in the character theory of finite groups. Although a complete answer to this question is yet to be found, many restrictions on the structure of the prime graph $\Delta(G)$ have been obtained. For example, it is known that $\Delta(G)$ has at most three connected components and if $\Delta(G)$ is connected, then its diameter is bounded above by three. (See [9, Theorems 6.4, 6.5].) For finite solvable groups, Pálfy's condition [14] asserts that given any three

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distinct primes in $\rho(G)$, there is always an edge connecting two primes among those primes. This condition is very useful in determining which graphs can occur as the prime graphs of finite solvable groups. In particular, this condition implies that if G is finite solvable, then $\Delta(G)$ has at most two connected components and if $\Delta(G)$ has exactly two connected components, then each component is complete. Unfortunately, this condition does not hold true in general. Nevertheless, it was proved in [13] that if $\pi \subseteq \rho(G)$ with $|\pi| \geqslant 4$, then there is an edge connecting two distinct primes in π.

The main purpose of this paper is to classify all k-regular graphs which can occur as $\Delta(G)$ for some finite group G, for $0 \leqslant k \leqslant 3$. Recall that a graph $\mathscr{G}$ is called k-regular for some integer $k \geqslant 0$, if every vertex of $\mathscr{G}$ has the same degree k. Combining results in [11,15], one can easily classify all k-regular prime graphs for $0 \leqslant k \leqslant 2$. (See Proposition 2.7 in Section 2.) In particular, if $\Delta(G)$ is 2-regular, then $\Delta(G)$ is a triangle or a square. For 3-regular graphs, we obtain the following result.

Theorem A. *The prime graph $\Delta(G)$ of a finite group G is 3-regular if and only if it is a complete graph with four vertices.*

Obviously, if $\Delta(G)$ is a complete graph with four vertices, then it is 3-regular. Therefore, we mainly focus on the 'only if' part. There are examples of both solvable and nonsolvable groups whose prime graphs are complete graphs with four vertices. For nonsolvable groups, we can simply take $G \cong \mathrm{A}_7$, the alternating group of degree 7. For solvable groups, we can take G to be a direct product of two solvable groups H and K, where both $\Delta(H)$ and $\Delta(K)$ are complete graphs with two vertices and $\rho(G) \cap \rho(H)$ is empty.

We mention that an analogous result for conjugacy class sizes was obtained by Bianchi et al. in [2], where the authors proved that the common-divisor graph $\Gamma(G)$, defined on the set of non-central conjugacy class sizes of a finite group G, is 3-regular if and only if it is a complete graph with four vertices and they conjectured that $\Gamma(G)$ is a k-regular graph if and only if it is a complete graph with $k+1$ vertices. Recently, this conjecture has been proved in [1].

The paper is organized as follows. In Section 2, we obtain an upper bound for the number of vertices of the prime graph $\Delta(G)$ of a finite group G in terms of the maximal degree d and the independent number of $\Delta(G)$ under the assumption that $\Delta(G)$ contains no subgraph isomorphic to a complete graph with $d+1$ vertices. (See Corollary 2.5.) This result may be useful in studying the prime graphs with bounded degrees. In Section 3, we prove Theorem A for solvable groups. This is achieved in Theorem 3.2. Section 4 is devoted to proving Theorem A for nonsolvable groups. This is the main part of the paper. Finally, in the last section, for each even integer $k \geqslant 2$, we construct a finite solvable group whose prime graph is k-regular with $k+2$ vertices.

All groups in this paper are assumed to be finite, all characters are complex characters and all graphs are finite, simple, undirected graphs (no loop nor multiple edge). We refer to [8] for the notation of character theory of finite groups and to [3] for terminology in graph theory. For an integer n, we write $\pi(n)$ for the set of all prime divisors of n. We write $\pi(G)$ instead of $\pi(|G|)$ for the set of all prime divisors of $|G|$. If $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$, then the inertia group of θ in G is denoted by $I_G(\theta)$. Finally, we write $\mathrm{Irr}(G|\theta)$ for the set of all irreducible constituents of θ^G.

2. Prime graphs of groups

In this section, we recall some graph theoretic terminologies and some known results in both graph and group theories which will be needed in this paper. We begin with some basic definitions and results in graph theory.

Let $\mathscr{G} = (V, E)$ be a graph of order $n = |V|$ with vertex set V and edge set E. Let v be a vertex of $\mathscr{G}$. The *degree* of v is the number of edges of $\mathscr{G}$ incident to v. A vertex $v \in V$ is said to be an *odd vertex* if its degree is odd. The following elementary result, which is a consequence of the Hand–Shaking lemma, is well known.

Lemma 2.1. *The number of odd vertices in a graph is even.*