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On subgroups of finite index in branch groups

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ABSTRACT

We give a structural description of the normal subgroups of subgroups of finite index in branch groups in terms of rigid stabilizers. This gives further insight into the structure lattices of branch groups introduced by the second author. We derive a condition concerning abstract commensurability of branch groups acting on the p-ary tree for any prime p.

1. Introduction

The class of branch groups was introduced by Grigorchuk in 1997, to provide a general framework for studying groups arising as counter-examples in a wide variety of contexts. This class also plays a natural role in the study of just infinite groups (see [9]). By now, the structure theory of branch groups is quite well developed; see for example [1,3,9]. Among the remarkable properties of arbitrary branch groups is a striking result proved by Grigorchuk describing their non-trivial normal subgroups; it has the consequence that every proper quotient of a branch group is virtually abelian. The definition of branch groups (given below) shows that some subgroups of finite index (for example, subgroups $\mathrm{rist}_G(n)$ with $n > 0$) do not share this property. However, our main theorem, Theorem 1.2 below, gives a reasonably precise description of normal subgroups of subgroups of finite index in branch groups.

Branch groups are defined in terms of their action on a specific type of tree. Let $(m_n)_{n\geqslant 0}$ be a sequence of integers with $m_n \geqslant 2$ for each n. The rooted tree of type (m_n) is a tree T with a vertex v_0 (called the root vertex) of valency m_0, such that every vertex at a distance $n \geqslant 1$ from v_0 has valency $m_n + 1$. The distance of a vertex v from v_0 is called the *level* of v, and the set L_n of vertices of level n is called the nth *layer* of T. Each vertex v of level r is the root of a rooted subtree T_v of type $(m_n)_{n\geqslant r}$. We picture T with the root at the top and with m_n edges descending from each vertex of

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level n. Therefore we call the vertices below a vertex v the *descendants* of v. If $m_n = d$ for every n, we say that T is a d-ary tree.

Now suppose that G is a group of tree automorphisms of T fixing v_0. For each vertex v write $\mathrm{rist}_G(v)$ for the subgroup of elements of G that fix all vertices outside T_v, and for each $n \geqslant 0$ write $\mathrm{rist}_G(n)$ for the direct product $\langle \mathrm{rist}_G(v) \mid v \in L_n \rangle$. We also write $\mathrm{rist}_G(X) = \prod_{x \in X} \mathrm{rist}_G(x)$ for each subset X of L_n. The group G is said to be a *branch group* on T if the following two conditions hold for each $n \geqslant 0$:

(i) G acts transitively on L_n;
(ii) $|G : \mathrm{rist}_G(n)|$ is finite.

Notice that the transitive action of G on all layers of T implies that G is infinite. Moreover G is evidently residually finite. We shall use these facts throughout the paper without further mention.

Branch groups are subject to strong restrictions. The proof of [4, Lemma 2] shows that they have no non-trivial virtually abelian normal subgroups and in [3, Theorem 4] the following description of normal subgroups is given:

Theorem 1.1. *Suppose that G is a branch group acting on a tree T and let $K \lhd G$ with $K \neq 1$. Then K contains the derived subgroup $\mathrm{rist}_G(n)'$ of $\mathrm{rist}_G(n)$ for some integer n.*

We shall prove the following result for normal subgroups of subgroups of finite index:

Theorem 1.2. *Suppose that G is a branch group acting on a tree T, let H be a subgroup of finite index and let $K \lhd H$. Then for all sufficiently large integers n there is a union X of H-orbits in L_n such that*

$$K \cap \mathrm{rist}_G(n)' = \mathrm{rist}_G(X)'. \tag{1}$$

More precisely,

$$\mathrm{rist}_G(X)' \leqslant K \quad \text{and} \quad K \cap \mathrm{rist}_G(L_n \setminus X) = \big[K, \mathrm{rist}_G(L_n \setminus X)\big] = 1. \tag{2}$$

The second part of (2) above is simply the statement that the subgroups K and $\mathrm{rist}_G(L_n \setminus X)$ generate their direct product in G.

It will follow easily from this result that for any subgroup H of finite index of a branch group, the number of infinite H-invariant direct factors of a normal subgroup of H is bounded by $|G : H|$. These results allow us to establish in Section 3 a necessary condition for direct products of certain branch groups acting on the p-ary tree to be (abstractly) commensurable, based on the number of direct factors. This complements a result of the first author [2] concerning the Gupta–Sidki 3-group.

We sketch another application. The structure lattice $\mathcal{L}$ and structure graph $\mathcal{T}$ of a branch group G have been defined and studied in [4,9]. They depend only on the structure of G as an abstract group and not on its action on the tree T. The structure lattice $\mathcal{L}$ is a Boolean lattice obtained from the set of all subgroups of G which have finitely many conjugates by declaring two such subgroups equivalent if their centralizers in G coincide. The structure graph $\mathcal{T}$ has as vertices the equivalence classes in $\mathcal{L}$ corresponding to *basal* subgroups. These are the subgroups whose finitely many conjugates generate their direct product (in particular, $\mathrm{rist}_G(v)$ is a basal subgroup for each $v \in T$). Defining the edges of $\mathcal{T}$ is slightly more technical, and we refer the reader to [4]. In that paper it is also shown that in many important cases $\mathcal{T}$ is in fact a tree and is even isomorphic to T. The action of G on its subgroup lattice induces an action by tree automorphisms on $\mathcal{T}$; thus we can find within the subgroup structure of G the tree on which G acts as a branch group and the action itself. Theorem 1.2 gives another approach to these objects. Let $\mathcal{L}_0$ be the family of subgroups of the form $\mathrm{rist}_G(X)$ with X an H-invariant subset of a layer of T for some subgroup $H \leqslant_{\mathrm{f}} G$. For such subsets X_1, X_2 write $\mathrm{rist}_G(X_1) \sim \mathrm{rist}_G(X_2)$ if $X_2 \subseteq L_n$ consists of all descendants of X_1 in L_n (or vice-versa). Then $\sim$ is an equivalence relation