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Hopf actions on filtered regular algebras

K. Chan [a], C. Walton [b,*], Y.H. Wang [c], J.J. Zhang [a]

[a] Department of Mathematics, Box 354350, University of Washington, Seattle, WA 98195, USA
[b] Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
[c] Department of Applied Mathematics, Shanghai University of Finance and Economics, Shanghai 200433, China

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ABSTRACT

We study finite dimensional Hopf algebra actions on so-called filtered Artin–Schelter regular algebras of dimension n, particularly on those of dimension 2. The first Weyl algebra is an example of such an algebra with $n = 2$, for instance. Results on the Gorenstein condition and on the global dimension of the corresponding fixed subrings are also provided.

0. Introduction

The main motivation for this paper (as well as for [4,3]) is to classify all finite dimensional Hopf algebras which act on a given algebra R. By understanding the Hopf algebras H which act on R, we can further study other structures related to R, such as the fixed ring R^H and the smash product $R \# H$. The prototype of this problem is classical: the classification of finite subgroups G of $\mathrm{SL}_2(\mathbb{C})$ (that act faithfully on the polynomial ring $\mathbb{C}[u,v]$) prompted the connection between the McKay quiver of G and the geometric features of the plane quotient singularity $\mathrm{Spec}(\mathbb{C}[u,v]^G)$. In our setting, the algebra R is allowed to be noncommutative and the Hopf algebras are allowed to be noncocommutative. More precisely, we study finite dimensional Hopf algebra actions on *filtered Artin–Schelter regular*

* Corresponding author.
E-mail addresses: kenhchan@math.washington.edu (K. Chan), notlaw@math.mit.edu (C. Walton), yhw@mail.shufe.edu.cn (Y.H. Wang), zhang@math.washington.edu (J.J. Zhang).

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algebras of dimension d. These are filtered algebras whose associated graded algebras are Artin–Schelter regular algebras of global dimension d. Our emphasis will be on the case of dimension 2.

Here, we assume that the base field k is algebraically closed of characteristic zero, unless otherwise stated. Examples of filtered Artin–Schelter regular algebras of dimension 2 include the first Weyl algebra $A_1(k) = k\langle u, v\rangle/(vu - uv - 1)$, quantum Weyl algebras $k\langle u, v\rangle/(vu - quv - 1)$ for some $q \in k^\times$, and other deformations of Artin–Schelter regular algebras of dimension 2.

The invariant theory of $A_1(k)$ by finite groups is already interesting. For example, the fixed subrings of $A_1(k)$ by finite groups actions are completely classified and studied by Alev, Hodges and Velez in [1]. Thus, it is natural to ask if there are any non-trivial finite dimensional Hopf algebra (H-)actions on the first Weyl algebra. By a "non-trivial" H-action, we mean that H is neither commutative nor cocommutative Hopf algebra (or neither a dual of a group algebra nor group algebra, respectively). We give a negative answer to this question in Theorem 0.1 below.

Recall that a left H-module M is called *inner-faithful* if $IM \neq 0$ for any nonzero Hopf ideal I of H. Let N be a right H-comodule with comodule structure map $\rho : N \to N \otimes H$. We say that this coaction is inner-faithful if $\rho(N) \nsubseteq N \otimes H'$ for any proper Hopf subalgebra $H' \subsetneq H$ [4, Definition 1.2]. We say that *a Hopf algebra H (co)acts on an algebra R* if R is a left H-(co)module algebra. Moreover, if the H-(co)module R is inner-faithful, then we say that H (co)acts on R *inner-faithfully*.

Theorem 0.1. *Let R be a non-PI filtered Artin–Schelter regular algebra of dimension 2 and let H be a finite dimensional Hopf algebra acting on R inner-faithfully. If the H-action preserves the filtration of R, then H is a group algebra.*

In particular, if R is the first Weyl algebra $A_1(k)$, then H is a group algebra.

Theorem 0.1 can be viewed as an extension of [4, Theorem 5.10] from the non-PI graded case to the non-PI filtered case. Most results in this work concern actions on non-PI AS regular algebras. In particular, combining Theorem 0.1 with [1, Proposition, p. 84], one classifies all finite dimensional Hopf algebras acting inner-faithfully on $A_1(k)$ with respect to the standard filtration (Corollary 5.7). Similarly, all finite dimensional Hopf algebras actions on the quantum Weyl algebras $k\langle u, v\rangle/(vu - quv - 1)$, for q not a root of unity, are classified (Corollary 5.8(a)). On the other hand, if R is PI filtered AS regular, then there are many interesting finite dimensional Hopf algebras (which are not group algebras) which act on R; see Examples 1.4 and 3.4 for instance.

Regarding the higher dimensional Weyl algebras, it is natural to ask the following question.

Question 0.2. Let $A_n(k)$ be the n-th Weyl algebra and let H be a finite dimensional Hopf algebra acting on $A_n(k)$ inner-faithfully. Is then H a group algebra?

If we assume that the H-action is filtration preserving, then the answer is yes if $n = 1$ (Theorem 0.1) or if H is pointed (Theorem 0.3).

Theorem 0.3. *Let H be a finite dimensional Hopf algebra acting on the n-th Weyl algebra $A_n(k)$ inner-faithfully which preserves the standard filtration of $A_n(k)$. Then H is semisimple. If, in addition, H is pointed, then H is a group algebra.*

In the setting of H-actions on graded algebras, we have the following result of Kirkman, Kuzmanovich and Zhang. Suppose that H is a semisimple finite dimensional Hopf algebra and R is an AS regular algebra. If H acts on R preserving the grading with trivial homological determinant, then the fixed subring R^H is AS Gorenstein [7, Theorem 0.1]. We can obtain a filtered analogue of the above by considering the induced H-action on $\mathrm{gr}_F R$.

Theorem 0.4. *Let R be a filtered AS regular algebra of dimension 2 and let H be a semisimple Hopf algebra acting on R. If the H-action is not graded and it preserves the filtration of R, then the fixed subring R^H is filtered AS Gorenstein.*