Journal of Algebra 397 (2014) 91–110

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

# Inner inverses and inner annihilators in rings

T.Y. Lam [a], Pace P. Nielsen [b,*]

[a] *Department of Mathematics, University of California, Berkeley, CA 94720, United States*
[b] *Department of Mathematics, Brigham Young University, Provo, UT 84602, United States*

ARTICLE INFO

*Article history:*
Received 4 March 2013
Available online 21 September 2013
Communicated by Changchang Xi

*MSC:*
primary 16E50
secondary 16D70, 16E20, 16U80

*Keywords:*
Jacobson pair
Inner annihilator
Inner inverse
Reflexive inverse
Group inverse
Von Neumann regular
Unit-regular
Strongly regular

ABSTRACT

For any ring element $\alpha \in R$, we study the group of inner annihilators $\mathrm{IAnn}(\alpha) = \{p \in R\colon\ \alpha p \alpha = 0\}$ and the set $I(\alpha)$ of inner inverses of $\alpha$. For any Jacobson pair $\alpha = 1 - ab$ and $\beta = 1 - ba$, the groups $A = \mathrm{IAnn}(\alpha)$ and $B = \mathrm{IAnn}(\beta)$ are shown to be equipotent, and $A \oplus C$ is shown to be group isomorphic to $B \oplus C$ where $C = \mathrm{Ann}_\ell(\alpha) \oplus \mathrm{Ann}_r(\alpha)$. In the case where $\alpha$ is (von Neumann) regular, we show further that $A \cong B$ as groups. For any Jacobson pair $\{\alpha, \beta\}$, a "new Jacobson map" $\Phi : I(\alpha) \to I(\beta)$ is constructed that is a semigroup homomorphism with respect to the von Neumann product, and preserves units, reflexive inverses and commuting inner inverses. In particular, for any abelian ring $R$, $\Phi$ is a semigroup isomorphism between $I(\alpha)$ and $I(\beta)$. As a byproduct of our methods, we also show that a ring $R$ satisfies internal cancellation iff every Jacobson pair of regular elements are equivalent over $R$. In particular, the latter property holds for many rings, including semilocal rings, unit-regular rings, strongly $\pi$-regular rings, and finite von Neumann algebras.



## 1. Introduction

Two elements $\alpha$, $\beta$ in a ring $R$ are said to form a *Jacobson pair* if there exist elements $a, b \in R$ such that $\alpha = 1 - ab$ and $\beta = 1 - ba$. For such a pair, "Jacobson's Lemma" is the statement that if $\alpha$ is invertible with inverse $s$, then $\beta$ is invertible with inverse $1 + bsa$. This leads easily to a proof of the similar fact that if $\alpha$ is regular (in the sense of von Neumann) with inner inverse $s$, then $\beta$ is also regular with inner inverse $1 + bsa$. However, the standard "Jacobson map" $s \mapsto 1 + bsa$ from the inner inverses of $\alpha$ to those of $\beta$ is usually neither injective nor surjective, fails to take units to units (if there are unit inner inverses), or group inverses to group inverses (if those exist).

* Corresponding author.
*E-mail addresses:* lam@math.berkeley.edu (T.Y. Lam), pace@math.byu.edu (P.P. Nielsen).

0021-8693/$ – see front matter 
http://dx.doi.org/10.1016/j.jalgebra.2013.08.023

In this paper, we define for every $\alpha \in R$ the set $\mathrm{IAnn}(\alpha) = \{p \in R\colon\ \alpha p \alpha = 0\}$ (consisting of the "inner annihilators" of $\alpha$). In case $\alpha$ is regular with an inner inverse $s$, the set of *all* inner inverses of $\alpha$, denoted by $I(\alpha)$, is given by $s + \mathrm{IAnn}(\alpha)$. So we expect $\mathrm{IAnn}(\alpha)$ to be a useful object to study in general, even when $\alpha$ is not regular. A search of the literature did not turn up very many results on $\mathrm{IAnn}(\alpha)$ and $I(\alpha)$, although some work has been done by Hartwig and Luh [7] on the set of *unit* inner inverses of unit-regular elements in a ring $R$. It is hoped that, by contributing a number of new results on $\mathrm{IAnn}(\alpha)$ and $I(\alpha)$, the present paper will draw interest on these objects from the ring theory community, and from algebraists in general.

In Section 2, we initiate the study of the abelian group $\mathrm{IAnn}(\alpha)$ in a ring $R$. Various motivating examples are given, with and without assuming that $\alpha$ is regular. In Section 3, we prove several theorems on ring elements with only a finite number of inner inverses or unit inner inverses that did not seem to have been noted before in the literature. In the second half of Section 3, we also formally introduce the two Hartwig–Luh self-maps $\varphi$ and $\varphi'$ on $I(\alpha)$ for any regular element $\alpha \in R$. These maps will play a substantial role in the subsequent study (in Sections 6–7) of mappings from $I(\alpha)$ to $I(\beta)$ for any Jacobson pair $\{\alpha, \beta\}$. There has been considerable recent interest on Jacobson pairs, mainly in connection with the study of generalized inverses (e.g. Drazin inverses) in rings; see, e.g., [2,4,17,18,21]. Indeed, it was this series of papers which aroused our own interest in the structure of the sets $\mathrm{IAnn}(\alpha)$ and $I(\alpha)$, and their relationship with $\mathrm{IAnn}(\beta)$ and $I(\beta)$ in case $\alpha$ and $\beta$ form a Jacobson pair.

In Section 4, we make our first analysis of $\mathrm{IAnn}(\alpha)$. By considering its subgroups $\mathrm{Ann}_\ell(\alpha)$ and $\mathrm{Ann}_r(\alpha)$ along with their sum and intersection, we prove the following result over any ring.

**Theorem A.** *For any Jacobson pair $\{\alpha, \beta\}$, the groups $\mathrm{IAnn}(\alpha)$ and $\mathrm{IAnn}(\beta)$ are always equipotent. Moreover, these two groups become isomorphic after "adding" the direct sum $\mathrm{Ann}_\ell(\alpha) \oplus \mathrm{Ann}_r(\alpha)$.*

In Section 5, we address the case of *regular* Jacobson pairs $\{\alpha, \beta\}$ (where $\alpha$, and hence $\beta$, is regular). In this more tractable case, we prove the following sharper version of Theorem A, again over any ring.

**Theorem B.** *If $\{\alpha, \beta\}$ is a regular Jacobson pair, then $\mathrm{IAnn}(\alpha) \cong \mathrm{IAnn}(\beta)$ as groups.*

While we are able to prove the existence of a group isomorphism in the setting of Theorem B, it seems difficult to construct an explicit *canonical* isomorphism from $\mathrm{IAnn}(\alpha)$ to $\mathrm{IAnn}(\beta)$. In the hope of solving this problem on the level of inner inverses (instead of inner annihilators), we turn our efforts to the alternative problem of finding "new Jacobson maps" $I(\alpha) \to I(\beta)$ that have better properties than the original Jacobson map (which sends any $s \in I(\alpha)$ to $1 + bsa \in I(\beta)$). Toward this goal, we prove the following main result in Sections 4–7.

**Theorem C.** *For any regular Jacobson pair $\{\alpha, \beta\}$ in a ring, the new Jacobson map $\Phi : I(\alpha) \to I(\beta)$ defined by*

$$\Phi(s) = 1 + bsa - b(1 - s\alpha)(1 - \alpha s)a \quad \bigl(\text{for any } s \in I(\alpha)\bigr) \tag{†}$$

*has the following properties:*

1. *$\Phi$ preserves and reflects units; that is, for $s \in I(\alpha)$, $s$ is a unit iff $\Phi(s)$ is a unit.*
2. *$\Phi$ preserves and reflects regular elements and unit-regular elements.*
3. *$\Phi$ is a semigroup homomorphism with respect to the von Neumann products in $I(\alpha)$ and $I(\beta)$. (In $I(\alpha)$, the von Neumann product is defined by $x * y = x\alpha y$, and similarly for $I(\beta)$.)*
4. *$\Phi$ preserves and reflects reflexive inverses. (A reflexive inverse of $\alpha$ is an element $s \in I(\alpha)$ such that $s = s\alpha s$, and similarly for $\beta$.)*
5. *$\Phi$ restricts to a bijection between the "commuting inner inverses" of $\alpha$ and those of $\beta$. Furthermore, the corresponding map from the commuting inner inverses of $\beta$ to those of $\alpha$ is the inverse map.*