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Modular representations of the special linear groups with small weight multiplicities

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ABSTRACT

We classify irreducible representations of the special linear groups in positive characteristic with small weight multiplicities with respect to the group rank and give estimates for the maximal weight multiplicities. For the natural embeddings of the classical groups, inductive systems of representations with totally bounded weight multiplicities are classified. An analogue of the Steinberg tensor product theorem for arbitrary indecomposable inductive systems for such embeddings is proved.

1. Introduction

In what follows K is an algebraically closed field of characteristic $p > 0$; G_n is a classical algebraic group of rank n over K; $\operatorname{Irr} G_n$ is the set of all rational irreducible representations (or simple modules) of G_n up to equivalence, $\operatorname{Irr}^p G_n \subset \operatorname{Irr} G_n$ is the subset of p-restricted ones; $\operatorname{Irr} M \subset \operatorname{Irr} G_n$ is the set of composition factors ofa module M (disregarding the multiplicities), $\omega(M)$ is the highest weight of a simple module M; $L(\omega)$ is the simple G_n-module with highest weight ω; $\omega_1^n, \ldots, \omega_n^n$ are the fundamental weights of G_n; $\omega_0^n = \omega_{n+1}^n = 0$ by convention. A weight $\sum_{i=1}^n a_i \omega_i^n$ is p-restricted if all

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$a_i < p$. By the *weight degree* of a module M we mean the maximal dimension of the weight subspaces in M, i.e.

$$\operatorname{wdeg} M = \max_{\mu \in \Lambda(M)} \dim M^{\mu}$$

where $\Lambda(M)$ is the set of weights of M. In particular, we say that M has a small weight degree if $\operatorname{wdeg} M$ is small with respect to n.

For the classical algebraic groups modular representations of weight degree 1 were classified in [19,25]. To state the result, first define the following sets of weights of the group $G_n = A_n(K)$, $B_n(K)$, $C_n(K)$, or $D_n(K)$:

$$\Omega_p\big(A_n(K)\big) = \big\{0, \omega_k^n, (p-1-a)\omega_k^n + a\omega_{k+1}^n \bigm| 0 \leqslant k \leqslant n,\ 0 \leqslant a \leqslant p-1\big\},$$
$$\Omega_p\big(B_n(K)\big) = \big\{0, \omega_1^n, \omega_n^n\big\},$$
$$\Omega_p\big(C_n(K)\big) = \left\{0, \omega_1^n, \frac{p-1}{2}\omega_n^n, \omega_{n-1}^n + \frac{p-3}{2}\omega_n^n\right\} \quad (p > 2),$$
$$\Omega_p\big(D_n(K)\big) = \big\{0, \omega_1^n, \omega_{n-1}^n, \omega_n^n\big\},$$
$$\Omega(G_n) = \left\{ \sum_{j=0}^{k} p^j \lambda_j \Bigm| k \geqslant 0,\ \lambda_j \in \Omega_p(G_n) \right\}.$$

Theorem 1.1. *(See [19, 6.1], [25, Proposition 2].) Let G_n be a classical algebraic group of rank $n \geqslant 4$ and let M be a rational simple G_n-module. Assume $p > 2$ for $G = B_n(K)$ or $C_n(K)$. Then $\operatorname{wdeg} M = 1$ if and only if $\omega(M) \in \Omega(G_n)$.*

Obviously, a simple module M is p-restricted with $\operatorname{wdeg} M = 1$ if and only if $\omega(M) \in \Omega_p(G_n)$. The $A_n(K)$-modules $L((p-1-a)\omega_k^n + a\omega_{k+1}^n)$ are truncated symmetric powers of the natural module [26, Proposition 1.2]. Thus, the only p-restricted modules of weight degree 1 for type A are the fundamental modules and truncated symmetric powers of the natural module. Recall that $B_n(K) \cong C_n(K)$ for $p = 2$ (as abstract groups). So we do not consider groups of type B_n in characteristic 2. For groups of type C_n in this case the description of irreducible modules of weight degree 1 is more involved (see details in Section 6).

In this paper we classify irreducible representations of the special linear groups of small weight degree. For other classical groups this was done by the authors earlier. In particular, it was shown that for these groups and odd p no irreducible modules M exist with $1 < \operatorname{wdeg} M < n - 7$.

Theorem 1.2. *(See [1, Theorem 1.1], [17, Theorem 1], [18, Theorem 1].) Let $n \geqslant 8$ and let $G_n = B_n(K)$, $C_n(K)$ or $D_n(K)$. Let M be a rational simple G_n-module with $\omega(M) \notin \Omega(G_n)$. Suppose that $p > 2$ for $G_n = B_n(K)$ or $C_n(K)$. Then $\operatorname{wdeg} M \geqslant n - 4 - [n]_4$ where $[n]_4$ is the residue of n modulo 4. In particular, $\operatorname{wdeg} M \geqslant n - 7$.*

The main case ($p > 2$ for $G_n = B_n(K)$ or $D_n(K)$ and $p > 7$ for $G_n = C_n(K)$) was settled in [1]; [17] deals with type D for $p = 2$; and [18] gives a new proof for type C for all p. For $G = C_n(K)$ and $p = 2$ a new exceptional series of modules with $\operatorname{wdeg} = 2^s$ appears (see details in Section 6).

Now assume that $G_n = A_n(K)$. Let $M \in \operatorname{Irr} G_n$, $\omega(M) = a_1\omega_1^n + \cdots + a_n\omega_n^n$, and M^* be the dual of M. Note that $\omega(M^*) = a_n\omega_1^n + a_{n-1}\omega_2^n + \cdots + a_1\omega_n^n$ and $\operatorname{wdeg} M = \operatorname{wdeg} M^*$. Define the *polynomial degree* of M as the polynomial degree of the corresponding polynomial representation of $GL_{n+1}(K)$, i.e.

$$\operatorname{pdeg} M = \sum_{k=1}^{n} k a_k. \tag{1}$$