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Branching laws for tensor modules over classical locally finite Lie algebras

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ABSTRACT

Let $\mathfrak{g}'$ and $\mathfrak{g}$ be isomorphic to any two of the Lie algebras $\mathrm{gl}(\infty), \mathrm{sl}(\infty), \mathrm{sp}(\infty)$, and $\mathrm{so}(\infty)$. Let M be a simple tensor $\mathfrak{g}$-module. We introduce the notion of an embedding $\mathfrak{g}' \subset \mathfrak{g}$ of general tensor type and derive branching laws for triples $\mathfrak{g}', \mathfrak{g}, M$, where $\mathfrak{g}' \subset \mathfrak{g}$ is an embedding of general tensor type. More precisely, since M is in general not semisimple as a $\mathfrak{g}'$-module, we determine the socle filtration of M over $\mathfrak{g}'$. Due to the description of embeddings of classical locally finite Lie algebras given by Dimitrov and Penkov in 2009, our results hold for all possible embeddings $\mathfrak{g}' \subset \mathfrak{g}$ unless $\mathfrak{g}' \cong \mathrm{gl}(\infty)$.

1. Introduction

Given an embedding $\mathfrak{g}' \subset \mathfrak{g}$ of two Lie algebras and a simple $\mathfrak{g}$-module M, the branching problem is to determine the structure of M as a $\mathfrak{g}'$-module. This is a classical problem in the theory of finite-dimensional Lie algebras. By Weyl's semisimplicity theorem, when $\mathfrak{g}'$ is finite-dimensional semisimple the branching problem reduces to finding the multiplicity of any simple $\mathfrak{g}'$-module M' as a direct summand of M. This is however not a simple task, due to the abundance of possible isomorphism classes of embeddings $\mathfrak{g}' \subset \mathfrak{g}$ (see [3]). Therefore, even for the classical series of Lie algebras explicit solutions of the branching problem are known only for specific cases. Such solutions are referred to as *branching laws* or *branching rules* and examples can be found in e.g. [10,5].

In this paper we consider the branching problem for the classical locally finite Lie algebras. These are the Lie algebras $\mathrm{gl}(\infty), \mathrm{sl}(\infty), \mathrm{sp}(\infty)$, and $\mathrm{so}(\infty)$, and they are defined as unions of the respective finite-dimensional Lie algebras under the upper-left corner inclusions. Here the situation is quite different from the finite-dimensional case. On the one hand, the description of the Lie algebra embeddings given in [2] is much simpler than the classical description of Dynkin in the finite-dimensional

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case. On the other hand, the modules of interest, called simple tensor modules, are in general not completely reducible over the subalgebra. Therefore, the branching problem involves more than just determining the multiplicities of all simple constituents. One has to determine a semisimple filtration of the given module over the subalgebra and it is a natural choice to work with the socle filtration. In this way, the goal of the present work is to solve the following branching problem. Given an embedding $\mathfrak{g}' \subset \mathfrak{g}$ of two classical locally finite Lie algebras and a simple tensor $\mathfrak{g}$-module M, find the socle filtration of M as a $\mathfrak{g}'$-module.

The structure of the paper is as follows. We start by giving some background on locally finite Lie algebras and by presenting some finite-dimensional branching laws which are used in the paper. Following the description of embeddings of classical locally finite Lie algebras given in [2], we introduce the notion of an embedding $\mathfrak{g}' \subset \mathfrak{g}$ of general tensor type. In the case $\mathfrak{g}' \ncong \mathrm{gl}(\infty)$ this notion describes all possible embeddings. One of the main results of the paper is Theorem 3.4 which shows that the branching problem for embeddings of general tensor type can be reduced to branching problems for embeddings of three simpler types. Then in Section 4 we determine explicitly the branching laws for these three types of embeddings in the case when $\mathfrak{g}', \mathfrak{g} \cong \mathrm{gl}(\infty)$ and M is any simple tensor $\mathfrak{g}$-module (Theorems 4.5, 4.9, 4.11). Since all other cases of embeddings follow the same ideas, we skip the proofs and list the end results in Tables A.1–A.5 in Appendix A.

2. Preliminaries

2.1. The classical locally finite Lie algebras

The ground field is $\mathbb{C}$. A countable-dimensional Lie algebra is called *locally finite* if every finite subset of $\mathfrak{g}$ is contained in a finite-dimensional subalgebra. Equivalently, $\mathfrak{g}$ is locally finite if it admits an exhaustion $\mathfrak{g} = \bigcup_{i\in\mathbb{Z}_{>0}} \mathfrak{g}_i$ where

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_i \subset \cdots$$

is a sequence of nested finite-dimensional Lie algebras. The classical locally finite Lie algebras $\mathrm{gl}(\infty)$, $\mathrm{sl}(\infty)$, $\mathrm{sp}(\infty)$, and $\mathrm{so}(\infty)$ are defined respectively as $\mathrm{gl}(\infty) = \bigcup_{i\in\mathbb{Z}_{>0}} \mathrm{gl}(i)$, $\mathrm{sl}(\infty) = \bigcup_{i\in\mathbb{Z}_{>0}} \mathrm{sl}(i)$, $\mathrm{sp}(\infty) = \bigcup_{i\in\mathbb{Z}_{>0}} \mathrm{sp}(2i)$, and $\mathrm{so}(\infty) = \bigcup_{i\in\mathbb{Z}_{>0}} \mathrm{so}(i)$ via the natural inclusions $\mathrm{gl}(i) \subset \mathrm{gl}(i+1)$, $\mathrm{sl}(i) \subset \mathrm{sl}(i+1)$, $\mathrm{sp}(2i) \subset \mathrm{sp}(2i+2)$, and $\mathrm{so}(i) \subset \mathrm{so}(i+1)$.

Next, we give an equivalent definition of the above four Lie algebras. Let V and V_* be countable-dimensional vector spaces over $\mathbb{C}$ and let $\langle\cdot,\cdot\rangle : V \times V_* \to \mathbb{C}$ be a non-degenerate bilinear pairing. The vector space $V \otimes V_*$ is endowed with the structure of an associative algebra such that

$$(v_1 \otimes w_1)(v_2 \otimes w_2) = \langle v_2, w_1\rangle v_1 \otimes w_2$$

where $v_1, v_2 \in V$ and $w_1, w_2 \in V_*$. We denote by $\mathrm{gl}(V, V_*)$ the Lie algebra arising from the associative algebra $V \otimes V_*$, and by $\mathrm{sl}(V, V_*)$ we denote its commutator subalgebra $[\mathrm{gl}(V, V_*), \mathrm{gl}(V, V_*)]$. If $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ is an antisymmetric non-degenerate bilinear form, we define the Lie algebra $\mathrm{gl}(V, V)$ as above by taking $V_* = V$. In this case $S^2(V)$, the second symmetric power of V, is a Lie subalgebra of $\mathrm{gl}(V, V)$ and we denote it by $\mathrm{sp}(V)$. Similarly, if $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ is a symmetric non-degenerate bilinear form, we again define $\mathrm{gl}(V, V)$ by taking $V_* = V$ and then $\bigwedge^2(V)$ is a Lie subalgebra of $\mathrm{gl}(V, V)$, which we denote by $\mathrm{so}(V)$.

The vector spaces V and V_* are naturally modules over the Lie algebras defined above, such that $(v_1 \otimes w_1) \cdot v_2 = \langle v_2, w_1\rangle v_1$ and $(v_2 \otimes w_2) \cdot w_1 = -\langle v_2, w_1\rangle w_2$ for any $v_1, v_2 \in V$ and $w_1, w_2 \in V_*$. We call them respectively the *natural* and the *conatural representations*. In the cases of $\mathrm{sp}(V)$ and $\mathrm{so}(V)$ we have $V = V_*$.

By a result of Mackey [7], there always exist dual bases $\{\xi_i\}_{i\in I}$ of V and $\{\xi_i^*\}_{i\in I}$ of V_* indexed by a countable set I, so that $\langle \xi_i, \xi_j^*\rangle = \delta_{ij}$. Using these bases, we can identify $\mathrm{gl}(V, V_*)$ with the Lie algebra $\mathrm{gl}(\infty)$. Similarly, $\mathrm{sl}(V, V_*) \cong \mathrm{sl}(\infty)$, $\mathrm{sp}(V) \cong \mathrm{sp}(\infty)$, and $\mathrm{so}(V) \cong \mathrm{so}(\infty)$.