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The Strong Factorial Conjecture

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ABSTRACT

In this paper, we present an unexpected link between the Factorial Conjecture [8] and Furter's Rigidity Conjecture [13]. The Factorial Conjecture in dimension m asserts that if a polynomial f in m variables X_i over $\mathbb{C}$ is such that $\mathcal{L}(f^k)=0$ for all $k\geqslant 1$, then $f=0$, where $\mathcal{L}$ is the $\mathbb{C}$-linear map from $\mathbb{C}[X_1,\dots,X_m]$ to $\mathbb{C}$ defined by $\mathcal{L}(X_1^{l_1}\cdots X_m^{l_m})=l_1!\cdots l_m!$. The Rigidity Conjecture asserts that a univariate polynomial map $a(X)$ with complex coefficients of degree at most $m+1$ such that $a(X)\equiv X \bmod X^2$, is equal to X if m consecutive coefficients of the formal inverse (for the composition) of $a(X)$ are zero.

1. Presentation

In Section 2, we recall the Factorial Conjecture from [8]. We give a natural stronger version of this conjecture which gives the title of this paper. We also recall the Rigidity Conjecture from [13]. We present an additive and a multiplicative inversion formula. We use the multiplicative one to prove that the Rigidity Conjecture is a very particular case of the Strong Factorial Conjecture (see Theorem 2.25). As an easy corollary we obtain a new case of the Factorial Conjecture (see Corollary 2.28). In Section 3, we study the Strong Factorial Conjecture in dimension 2. We give a new proof of the Rigidity Conjecture $R(2)$ (see Section 3.1) using the Zeilberger Algorithm (see [16]). We study the case of two monomials (see Section 3.2). In Section 4 (resp. Section 5) we shortly give some historical details about the origin of the Factorial Conjecture (resp. the Rigidity Conjecture).

2. The bridge

In this section, we fix a positive integer $m\in\mathbb{N}_+$. By $\mathbb{C}^{[m]}=\mathbb{C}[X_1,\dots,X_m]$, we denote the $\mathbb{C}$-algebra of polynomials in m variables over $\mathbb{C}$.

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2.1. The Strong Factorial Conjecture

We recall the definition of the *factorial map* (see [8, Definition 1.2]):

Definition 2.1. We denote by $\mathcal{L}:\mathbb{C}^{[m]}\to\mathbb{C}$ the linear map defined by

$$\mathcal{L}\big(X_1^{l_1}\cdots X_m^{l_m}\big)=l_1!\cdots l_m!\quad\text{for all } l_1,\dots,l_m\in\mathbb{N}.$$

Remark 2.2. Let $\sigma\in\mathfrak{S}_m$ be a permutation of the set $\{X_1,\dots,X_m\}$. If we extend σ to an automorphism $\tilde{\sigma}$ of the $\mathbb{C}$-algebra $\mathbb{C}^{[m]}$, then for all polynomials $f\in\mathbb{C}^{[m]}$, we have $\mathcal{L}(\tilde{\sigma}(f))=\mathcal{L}(f)$.

Remark 2.3. The linear map $\mathcal{L}$ is not compatible with the multiplication. Nevertheless, $\mathcal{L}(fg)=\mathcal{L}(f)\mathcal{L}(g)$ if $f,g\in\mathbb{C}^{[m]}$ are two polynomials such that there exists an $I\subset\{1,\dots,m\}$ such that $f\in\mathbb{C}[X_i\colon i\in I]$ and $g\in\mathbb{C}[X_i\colon i\notin I]$.

We recall the Factorial Conjecture (see [8, Conjecture 4.2]).

Conjecture 2.4 *(Factorial Conjecture FC(m)). For all $f\in\mathbb{C}^{[m]}$,*

$$(\forall k\in\mathbb{N}_+)\mathcal{L}\big(f^k\big)=0\quad\Rightarrow\quad f=0.$$

To state some partial results about this conjecture it is convenient to introduce the following notation:

Definition 2.5. We define the *factorial set* as the following subset of $\mathbb{C}^{[m]}$:

$$F^{[m]}=\big\{f\in\mathbb{C}^{[m]}\setminus\{0\};\,(\exists k\in\mathbb{N}_+)\mathcal{L}\big(f^k\big)\neq 0\big\}\cup\{0\}.$$

Remark 2.6. Let $f\in\mathbb{C}^{[m]}$ be a polynomial, we have $f\in F^{[m]}$ if and only if:

$$(\forall k\in\mathbb{N}_+)\mathcal{L}\big(f^k\big)=0\quad\Rightarrow\quad f=0.$$

In other words, the factorial set $F^{[m]}$ is the set of all polynomials satisfying the Factorial Conjecture $FC(m)$ and this conjecture is equivalent to $F^{[m]}=\mathbb{C}^{[m]}$.

To give a stronger version of this conjecture we introduce the following subsets of $\mathbb{C}^{[m]}$:

Definition 2.7. For all $n\in\mathbb{N}_+$, we consider the following subset of $\mathbb{C}^{[m]}$:

$$F_n^{[m]}=\big\{f\in\mathbb{C}^{[m]}\setminus\{0\};\,\big(\exists k\in\big\{n,\dots,n+\mathcal{N}(f)-1\big\}\big)\mathcal{L}\big(f^k\big)\neq 0\big\}\cup\{0\}$$

where $\mathcal{N}(f)$ denotes the number of (nonzero) monomials in f. We define *the strong factorial set* as:

$$F_\cap^{[m]}=\bigcap_{n\in\mathbb{N}_+}F_n^{[m]}.$$

Since, for all $n\in\mathbb{N}_+$, it's clear that $F_n^{[m]}\subset F^{[m]}$, the following conjecture is stronger than the Factorial Conjecture.