Journal of Algebra 397 (2014) 515–534

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Equivalent notions of normal quantum subgroups, compact quantum groups with properties *F* and *FD*, and other applications

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ARTICLE INFO

Article history:
Received 25 February 2013
Available online 2 October 2013
Communicated by Nicolás Andruskiewitsch

MSC:
primary 16T05, 46L87, 58B32
secondary 46L55, 46L89, 81R50, 81R60

Keywords:
Compact quantum groups
Woronowicz C^*-algebras
Hopf algebras
Non-commutative geometry

ABSTRACT

The notion of normal quantum subgroup introduced in algebraic context by Parshall and Wang when applied to compact quantum groups is shown to be equivalent to the notion of normal quantum subgroup introduced by the author. As applications, a quantum analog of the third fundamental isomorphism theorem for groups is obtained, which is used along with the equivalence theorem to obtain results on structure of quantum groups with property *F* and quantum groups with property *FD*. Other results on normal quantum subgroups for tensor products, free products and crossed products are also proved.

1. Introduction

The notion of normal quantum subgroup is an important and subtle concept in the theory of quantum groups. In purely algebraic context of Hopf algebras, B. Parshall and J. Wang [17] defined a notion of normal quantum subgroup using left and right adjoint coactions of the Hopf algebra on itself, which was further studied by other authors such as Schneider [20], Takeuchi [24], and Andruskiewitsch and Devoto [1]. Parshall and Wang noted that left normal quantum groups may not be right normal in general, and given a normal quantum subgroup in their sense, it is not known whether there exists an associated exact sequence, and moreover if an exact sequence exists, it may not be unique. These difficulties are peculiar phenomena of general Hopf algebras in purely algebraic context distinguishing Hopf algebras from groups. Other complications related to the notion of normal quantum

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0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2013.09.014

groups in purely algebraic context are included in [20]. In C^*-algebraic context, the author introduced [28] a notion of normal quantum subgroup of compact quantum groups using analytical properties of representation theory of compact quantum groups. It was not known whether these two notions of normality are equivalent when they are applied to the canonical dense Hopf $*$-algebras of quantum representative functions of compact quantum groups. In [33], the author's notion of normal quantum groups was used in an essential way to define the notion of simple compact quantum groups. It was also announced in [33] without proof that the above two notions of normality are equivalent for compact quantum groups (see remark (b) after Lemma 4.4 in [33]). As consequences, left normal and right normal defined in algebraic context by Parshall and Wang are also equivalent for compact quantum groups, and their normal quantum subgroups always give rise to a unique exact sequence. That is, the complications mentioned above in purely algebraic setting do not present themselves in the world of compact quantum groups. Such properties might be useful for formulating an appropriate notion of quantum groups in algebraic setting, which is still an open problem.

Other facts announced in [33] without proofs include general results on structure of compact quantum groups with property F (resp. property FD), where, roughly speaking, a compact quantum group G is said to have property F if its quantum function algebra A_G has the same property with respect to quotients by normal quantum subgroups as the function algebra of a compact group, and it is said to have property FD if its quantum function algebra has the same property with respect to quotients by normal quantum subgroups as the quantum function algebra of the dual of a discrete group. See Definition 4.2 below for precise definitions of these concepts and notation used above. Compact quantum groups with property F include all quantum groups obtained from compact Lie groups by deformation method, such as compact real form of the Drinfeld–Jimbo quantum groups and Rieffel's deformation, as well as most of the universal quantum groups constructed by the author except the universal unitary quantum groups $A_u(Q)$ (also called the free unitary quantum groups), cf. [33].

The purposes of this paper are to give complete proof of the equivalence of the two notions of normality mentioned above and give the following applications of this Equivalence Theorem on the structure of compact quantum groups.

(1) We establish a complete quantum analog of the Third Fundamental Isomorphism Theorem. This is the only one among the three fundamental isomorphism theorems that has a complete quantum analog without added conditions or restrictions. On the contrary, a surjection of compact quantum groups (i.e. inclusion of Woronowicz C^*-algebras) does not always give rise to a quantum analog of the First Fundamental Isomorphism Theorem, except in the special case where an exact sequence can be constructed, cf. [17,20,24,1] for this and other subtleties. Taking the example of the group C^*-algebra $A_G := C^*(F_2)$ of the free group F_2 on two generators, a Woronowicz C^*-subalgebra of A_G does not give rise to an exact sequence unless it is the group C^*-algebra of a normal subgroup of F_2. In addition, it is not clear at the moment how a quantum analog of the second fundamental isomorphism theorem can be formulated.

(2) Using the Equivalence Theorem and the quantum analog of the Third Fundamental Isomorphism Theorem, we show that quotient quantum groups of a compact quantum group with property F also have property F, and quantum subgroups of a compact quantum group with property FD also have property FD. We show that quotient quantum groups of a compact quantum group with property FD also have property FD provided G has the pullback property. The pullback property is the quantum group version of the group situation in which every subgroup of G/N is of the form H/N for some subgroup H of G containing N. We give an example to show not all compact quantum groups have the pullback property.

(3) We prove results on normal quantum subgroups for tensor products, free products and crossed products. Note that the free product construction has no place in the classical world of compact groups. It is a total quantum phenomenon.

An outline of the paper is as follows. In Section 2, we recall the algebraic notion of normal quantum subgroups in [17] and the analytical notion of normal quantum subgroups in [28] respectively. In Section 3, the equivalence of these two notions of normality is proved. In Section 4, as applications of the Equivalence Theorem, we prove the quantum analog of the Third Fundamental Isomorphism Theorem, and results on structure of compact quantum groups with property F and property FD. In