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Counting irreducible representations of the Heisenberg group over the integers of a quadratic number field

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ABSTRACT

We calculate the representation zeta function of the Heisenberg group over the integers of a quadratic number field. In general, the representation zeta function of a finitely generated torsion-free nilpotent group enumerates equivalence classes of representations, called twist-isoclasses. This calculation is based on an explicit description of a representative from each twist-isoclass. Our method of construction involves studying the eigenspace structure of the elements of the image of the representation and then picking a suitable basis for the underlying vector space.

1. Introduction

Let G be a finitely generated torsion-free nilpotent group. Let χ be a 1-dimensional complex representation and ρ an n-dimensional complex representation of G. We define the product $\chi \otimes \rho$ to be a *twist* of ρ. Two representations ρ and ρ_* are *twist-equivalent* if, for some 1-dimensional representation χ, $\chi \otimes \rho \cong \rho_*$. This twist-equivalence is an equivalence relation on the set of irreducible representations of G. In [18] Lubotzky and Magid call the equivalence classes *twist-isoclasses*. They also show that there are only finitely many irreducible n-dimensional complex representations up to twisting and that for each $n \in \mathbb{N}$ there is a finite quotient N of G such that each n-dimensional irreducible representation ρ of G is twist-equivalent to one that factors through N. Henceforth we call the complex representations of G simply representations. We denote the number of twist-isoclasses of irreducible representations of dimension n by $r_n(G)$ or r_n if no confusion will arise.

Consider the formal Dirichlet series

$$\zeta_G(s) = \sum_{n=1}^{\infty} r_n(G) n^{-s}. \tag{1}$$

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If G is a finitely generated torsion-free nilpotent group then by [22, Lemma 2.1], $\zeta_G(s)$ converges on a right half plane of $\mathbb{C}$, say D, where $D := \{s \in \mathbb{C} \mid \Re(s) > \alpha\}$ for some $\alpha \in \mathbb{R}$; we call $\zeta_G : D \to \mathbb{C}$ the *representation zeta function* of G. Note that in [23,16,22] this function is denoted $\zeta_G^{\mathrm{irr}}(s)$. For a prime p, define

$$\zeta_{G,p}(s) := \sum_{n=0}^{\infty} r_{p^n}(G) p^{-ns}, \tag{2}$$

to be the *p-local representation zeta function* of $\zeta_G(s)$. Since G is nilpotent, each finite quotient decomposes as a direct product of its Sylow-p subgroups. Since the irreducible representations of a direct product of finite groups are tensor products of irreducible representations of its factors, we have the Euler product $\zeta_G(s) = \prod_p \zeta_{G,p}(s)$ [22, Introduction]. Moreover, it was shown by Hrushovski and Martin [13, Theorem 8.4] that each p-local representation zeta function is a rational function in p^{-s}.

The idea of using zeta functions to study the representation growth of groups is motivated by subgroup growth, where one uses zeta functions to count finite index subgroups. The study of subgroup growth of finitely generated nilpotent groups by zeta functions was introduced in [11]. In that paper, the authors calculate the normal subgroup zeta function of the Heisenberg group over the ring of integers of a number field of degree at most two [11, Prop. 8.2]. These zeta functions were given, in part, in terms of the Dedekind zeta function of the associated number field. Research on subgroup zeta functions of nilpotent groups continued in papers such as [9,10,8,23].

One can study the growth rate of the sequence $r_n(G)$ without necessarily explicitly constructing the zeta function. We call this study *representation growth*. Representation growth has been used to study other classes of groups. We briefly mention some work done in this area. For the following groups, $r_n(G)$ counts irreducible representations, as opposed to twist-isoclasses as with nilpotent groups. The idea of using zeta functions to study representation growth was introduced in [24], in which Witten studies compact Lie groups. Later, representation growth was studied for S-arithmetic groups by Lubotzky and Martin [19] and, using the language of representation zeta functions, Larsen and Lubotzky [17]. Jaikin, in [14], studies representation zeta functions of compact p-adic analytic groups with property FAb. In [4] Avni et al. study representation zeta functions of compact p-adic analytic and arithmetic groups; see also the research announcement [3]. In [2] they prove a conjecture of Larsen and Lubotzky (see [17, Conjecture 1.5]). In [1] Avni shows that arithmetic groups in characteristic zero which satisfy the congruence subgroup property have representation zeta functions with rational abscissa of convergence. Bartholdi and de la Harpe study representation zeta functions of wreath products with finite groups [5]. In [15], Kassabov and Nikolov study representation growth of some profinite groups. Craven gives lower bounds for representation growth for profinite and pro-p groups [7].

Representation growth of finitely generated nilpotent groups has been studied in [23] by Voll and [22] by Stasinski and Voll. Very few examples of representation zeta functions of finitely generated nilpotent groups have appeared in the literature (see [22, Theorem B] and [13, Example 8.12]). Let

$$H(\mathbb{Z}) := \langle x, y, z \mid [x, y] = z,\ z \text{ is central} \rangle \tag{3}$$

and let $\zeta(s)$ be the Riemann zeta function. Note that $H(\mathbb{Z})$ is isomorphic to the group of 3×3 upper-unitriangular matrices over the rational integers; this is normally called the discrete Heisenberg group. Then, by [13, Example 8.12], $\zeta_{H(\mathbb{Z})}(s)$ is given by

$$\zeta_{H(\mathbb{Z})}(s) = \frac{\zeta(s-1)}{\zeta(s)} \tag{4}$$

(the coefficients $r_{p^n}(H(\mathbb{Z}))$ were originally calculated in [21, Theorem 5]). This has a simpler shape than the corresponding normal subgroup zeta function in [20, Chapter 15].