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Linearity for actions on vector groups

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ABSTRACT

Let k be an arbitrary field, let G be a (smooth) linear algebraic group over k, and let U be a vector group over k on which G acts by automorphisms of algebraic groups. The action of G on U is said to be *linear* if there is a G-equivariant isomorphism of algebraic groups $U \simeq \mathrm{Lie}(U)$.

Suppose that G is connected and that the unipotent radical of G is defined over k. If the G-module $\mathrm{Lie}(U)$ is simple, we show that the action of G on U is linear. If G acts by automorphisms on a connected, split unipotent group U, we deduce that U has a filtration by G-invariant closed subgroups for which the successive factors are vector groups with a linear action of G. When G is connected and the unipotent radical of G is defined and split over k, this verifies an assumption made in earlier work of the author on the existence of Levi factors.

On the other hand, for any field k of positive characteristic we show that if the category of representations of G is not semisimple, there is an action of G on a suitable vector group U which is not linear.

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1. Introduction

Let k be a field, and let G denote a linear algebraic group over k; otherwise said, G is a smooth affine group scheme of finite type over k. A vector group U is a linear algebraic group (over k) isomorphic to the product of (finitely many) copies of the additive group $\mathbf{G}_a$.

In this paper, we are interested in the action of G by algebraic group automorphisms on a vector group U. If the linear algebraic group G acts on any linear algebraic group H by algebraic group automorphisms, the induced action of G on the Lie algebra of H makes $\mathrm{Lie}(H)$ a G-module. We say that the action of G on U is *linear* if there is a G-equivariant isomorphism of algebraic groups $U \simeq \mathrm{Lie}(U)$.[2]

If k has characteristic 0, view U as a closed subgroup of $\mathrm{GL}(V)$ for some faithful finite dimensional U-module V. Then every vector in $\mathrm{Lie}(U)$ is a nilpotent endomorphism of V, and the exponential mapping $X \mapsto \exp(X)$ defines a G-equivariant isomorphism of algebraic groups $\mathrm{Lie}(U)_a \xrightarrow{\sim} U$. On the other hand, if k has characteristic $p > 0$, in Section 5, we give examples of non-linear actions of G whenever there are G-modules which are not completely reducible (in particular, for semisimple groups G). *Thus, our results are only interesting when k has characteristic $p > 0$, which we assume from now on.*

Our main result gives a sufficient condition for linearity of the action of G on U which holds under some hypothesis which we now discuss.

1.1. Assumptions on G

When the ground field k is imperfect, the geometric unipotent radical of G – i.e. the unipotent radical of $G_{/k_{\mathrm{alg}}}$ – may not arise by base-change from any subgroup of G – see e.g. [3, Example 1.1.3]. We are going to sidestep this issue here. Consider the conditions

(R) there is a subgroup $R \subset G$ such that $R_{/k_{\mathrm{alg}}}$ is the unipotent radical of $G_{/k_{\mathrm{alg}}}$, and
(RS) condition **(R)** holds and R is split over k.

Recall that a connected unipotent group U is *split* provided that there is a filtration

$$U = U^0 \supset U^1 \supset \cdots \supset U^r = 1$$

by closed normal subgroups for which each subquotient U^i/U^{i+1} is a *vector group*. When k is imperfect, there are so-called *wound* unipotent groups which are not split – see e.g. [3, Example B.2.3] or Example (2.2.4) below.

If **(R)** holds, we refer to the group $R \subset G$ as the unipotent radical of G. In the language used in [13], condition **(R)** means that the unipotent radical of G is defined over k, and **(RS)** means that the unipotent radical of G is defined and split over k. Observe that conditions **(R)** and **(RS)** are automatic for any G when k is perfect; see e.g. [13, 14.4.5(v) and 14.3.10]. If **(R)** holds, the quotient G/R is a (not necessarily connected) *reductive* algebraic group over k.

1.2. The main result: a condition for linearity

If the linear algebraic group G acts by group automorphisms on the vector group U, then $\mathrm{Lie}(U)$ is a G-module and hence a module for the identity component G^0 of G. The following condition for the linearity of the action of G on U will be obtained in Theorem (3.2.6):

[2] A finite dimensional k-vector space V may be viewed as a linear algebraic group – in fact, a vector group – in a natural way. In what follows, we will write V_a when we view V as an algebraic group. With this notation, the action of G on the vector group U is linear if there is a G-equivariant isomorphism $U \simeq \mathrm{Lie}(U)_a$ of algebraic groups.

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